SAMPLE EXAM #2

Business Calculus

SHOW ALL WORK – NO NEED TO SIMPLIFY ANSWERS

1. (20 points) Using differentials, approximate $\sqrt{4.1}$.

> **Solution:** Let $f(x) = \sqrt{x}$, let $x = 4$ and let $\Delta x = 4.1 - 4 = \frac{1}{10}$. Notice that $f'(x) = \dfrac{1}{2\sqrt{x}}$ and, using differentials,
>
> $$\sqrt{4.1} = f(4.1) \approx f(4) + \underbrace{f'(4)\Delta x}_{df} = \sqrt{4} + \frac{1}{2\sqrt{4}}\left(\frac{1}{10}\right) = \frac{81}{40}.$$

2. (20 points) Find $[x^x]'$.

> **Solution:**
>
> $$[x^x]' = \left[\left(e^{\ln(x)}\right)^x\right]' = \left[e^{x\ln(x)}\right]' = e^{x\ln(x)}\left(1\ln(x) + x\left(\frac{1}{x}\right)\right) = x^x\left(\ln(x) + 1\right).$$

3. (20 points) Find $\left[\log_{10}(x^2+1)\right]'$.

> **Solution:**
>
> $$\begin{aligned} \left[\log_{10}(x^2+1)\right]' &= \left[\frac{\ln(x^2+1)}{\ln((10)}\right]' \\ &= \frac{1}{\ln(10)}\left[\ln(z^2+1)\right]' \\ &= \frac{1}{\ln(10)}\left(\frac{1}{x^2+1}\right)2x = \frac{2x}{(x^2+1)\ln(10)}. \end{aligned}$$

4. (20 points) A stamp collection is worth \$1,200 and its value increases linearly at \$200 a year. If the prevailing interest rate remains constant at 8% per year, compounded continuously, when will it be most advantageous to sell the stamp collection? (Section 4.4, Problem 50)

Solution:

$$\begin{aligned} \text{Value} &= 1,200 + 200t \\ \text{Present Value} &= P(t) = (1.200 + 200t)e^{-0.08t} \\ 0 &= P'(t) = 200e^{-0.08t} + (1,200 + 200t)e^{-0.08t}(-0.08) \\ 0 &= 200 - 96 - 16t \\ t &= \frac{104}{16} = \frac{13}{2}. \end{aligned}$$

Evaluate the following:

5. (20 points) $\displaystyle\int_1^9 x^{3/2}\,dx.$

Solution:

$$\int_1^9 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_1^9 = \frac{2}{5}(243-1) = \frac{484}{5}.$$

6. (20 points) $\displaystyle\int \frac{x^2}{\sqrt{x^3-3}}\,dx.$

Solution: Let $u = x^3 - 3$. Then $\frac{du}{dx} = 3x^2$ and $dx = \dfrac{du}{3x^2}$. Substituting,

$$\int \frac{x^2}{\sqrt{u}}\frac{du}{3x^2} = \frac{1}{3}\int u^{-1/2}\,du = \frac{1}{3}\left(2u^{1/2}\right) + C = \frac{2}{3}\sqrt{x^3-3} + C.$$

7. (20 points) What are the dimensions of the largest box (in volume) one can make with \$100 if each square foot of top or bottom cost \$2, if each square foot of side cost \$1, and if the width equals the length?

Solution: Want: x and y where x = length = width and y = height.

Know:

$$\begin{aligned} V &= \text{volume} = x^2y \qquad \text{to be maxed} \\ C &= \text{cost} \\ &= \text{top} + \text{bottom} + 4\text{sides} \\ &= 2x^2 + 2x^2 + 4xy \qquad \text{the restraint} \end{aligned}$$

To Do: From the constraint one has

$$y = \frac{100 - 4x^2}{4x} = \frac{25 - x^2}{x}$$

Thus

$$V(x) = x^2\left(\frac{25 - x^2}{x}\right) = 25x - x^3.$$

Maximizing $V(x)$ for $0 \le x \le 5$ one sees

$$0 = V'(x) = 25 - 3x^2 \quad \Rightarrow \quad x = \frac{5}{\sqrt{3}} = \frac{5\sqrt{3}}{3}.$$

Since the max volume is not at 0 or 5 and since $V'(x)$ exists everywhere, the max volume dimensions are $x = \frac{5\sqrt{3}}{3}$ and $y = \frac{10\sqrt{3}}{3}$.

8. (20 points) If you throw a rock upward and one second later its velocity is 16 feet per second (upward), and two seconds later it is 32 feet high. How high will it go? Remember, acceleration due to gravity is -32 feet per second squared.

Solution:

$$\begin{aligned} v(t) &= \int a(t)\,dt = \int -32\,dt = -32t + C \\ & \quad 16 = v(1) = -32 + C \quad \Rightarrow \quad C = 48 \\ v(t) &= -32t + 48 \\ p(t) &= \int v(t)\,dt = \int -32t + 48\,dt = -16t^2 + 48t + K \\ & \quad 32 = p(2) = -64 + 96 \quad \Rightarrow \quad K = 0 \\ p(t) &= -16t^2 + 48t. \end{aligned}$$

Let t_0 be when rock is at it highest. Then $0 = v(t_0) = -32t_0 + 48$ or $t_0 = 3/2$. Thus, at its highest point the rock is $p(3/2) = -16(3/2)^2 + 48(3/2) = 36$ feet high.

9. (20 points) What is the present value of an annuity that pays at a rate of \$10,000 a year for ten years worth if the prevailing interest rate remains constant at 4% a year, compounded continuously? (Section 5.5, Problem 30)

Solution:

$$\begin{aligned} \text{Present Value} &= \int_0^{10} 10,000e^{-0.04t}\,dt \\ &= -10,000\left(25e^{-t/25}\right)\Big]_0^{10} \\ &= -250,000(e^{-2/5} - 1) = 82,420. \end{aligned}$$

10. (20 points) Assume $D(x) = 65 - x^2$ and $S(x) = \frac{x^2}{3} + 2x + 5$. Find the equilibrium price, p_e and the consumers' surplus. (Section 5.5, Problem 16)

Solution:

$$\begin{aligned} 65 - x_e^2 = D(x_e) &= S(x_e) = \frac{x_e^2}{3} + 2x_e + 5 \\ \frac{4}{3}x_e^2 + 2x_e - 60 &= 0 \\ 4x_e^2 + 6x_e - 180 &= 0 \\ 2x_e^2 + 3x_e - 90 &= 0 \end{aligned}$$

Using the quadratic formula one has $x_e = \frac{-3 \pm \sqrt{3^2 - 4(2)(-90)}}{2(2)} = \frac{-3 \pm 27}{2(2)}$, i.e., x_e equals $-\frac{15}{2}$ or 6. Since x_e is positive, $x_e = 6$ and $p_e = D(6) = S(6) = 29$. Furthermore

$$\begin{aligned} \text{Consumers's Surplus} &= \int_0^{x_e} D(x) - p_e \, dx \\ &= \int_0^6 (65 - x^2) - 29 \, dx = 36x - \frac{1}{3}x^3 \Big]_0^6 = 144. \end{aligned}$$