CHAIN RULE: A Plausibility Argument

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Suppose Jack is $p(t)$ miles down the road at time t hours. Suppose also that Jill films Jack's trip from a stationary point starting at time 0. Her camera uses one click (a unit of length) of film per hour. Later, after eating a peanut butter and jelly sandwich, Jill plays back the tape of Jack's trip. Suppose the projector has shown $u(t)$ clicks of film after t hours of viewing. How fast is Jack going, on the screen, at t hours into the viewing? One can go about answering this question two ways.

One way of answering this question is to notice that after t hours into the viewing, Jack (on the screen) is $u(t)$ hours into his trip. Jack's speed at time $u(t)$ was $p'(u(t))$. However since the film is moving by at $u'(t)$ clicks per hour, Jack appears to be going at $u'(t)$ times his actual speed. Thus Jack's screen speed, after Jill has watched the tape t hours, will be $p'(u(t))u'(t)$.

Another way of looking at the problem is that Jack is $(p \circ u)(t)$ miles down the road in the film after t hours of watching. Thus his screen speed is $(p \circ u)'(t)$.

Since both ways of looking at this problem are correct, one sees that Jack's screen speed, t hours into the viewing, is

$$(p \circ u)'(t) = p'(u(t))u'(t).$$

This is exactly the chain rule.

Theorem 0.1 (Chain Rule) *If $g(\cdot)$ is differentiable at x and $f(\cdot)$ is differentiable at $g(x)$ then $h(\cdot) \stackrel{\text{def}}{=} f \circ g(\cdot)$ is differentiable at x with $h'(x) = f'(g(x))g'(x)$.*