STRUCTURE AND MOVING AVERAGE REPRESENTATION FOR MULTIDIMENSIONAL STRONGLY HARMONIZABLE PROCESSES

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Abstract

The classical moving average representation of stationary processes is generalized to moving average representations for discrete and continuous multidimensional strongly harmonizable processes. Necessary and sufficient conditions on covariance functions are given for the existence of such moving average representations. Also the structure theory of strongly harmonizable processes is given in considerable detail.

1 OVERVIEW

The purpose of this paper is to obtain the moving average representation for multidimensional strongly harmonizable processes. This includes stationary processes as a proper subset. The former is a proper subset of all bounded continuous processes and they are a natural extension of stationary processes. This class of processes, whose study is amiable to Fourier analytic methods, is of interest for applications such as prediction theory, filtering problems and others.

In section 2 a brief account of spectral representation of stationary and harmonizable processes is given. A discussion of some necessary and sufficient conditions for a bimeasure to be a spectral measure follows.

A new proof of the existence of a Hilbert Space isomorphism between the time domain and the frequency domain for strongly harmonizable processes is presented in section 3 and used later on.

In section 4 attention is given to those processes whose covariance functions, $r(\cdot,\cdot)$, can be written as $r(s,t) = G(s)G^*(t)$ where $G(\cdot)$ is a matrix valued function. These are called "splitting processes" and they correspond to a finite dimensional space of observables, and form the building blocks of more general processes. The density of the class of harmonizable splitting processes in the class of all harmonizable processes is discussed.

A notion of a virile representation of the covariance function of a harmonizable process is introduced and a corresponding concept of rank is discussed in section 5. Some examples are presented.

Section 6 introduces virile moving averages and gives a characterization of strongly harmonizable processes with a virile covariance representation of full rank. This result is a generalization of the classical case treated by Y. Rozanov extended to a new class of moving average representations.

Section 7 is devoted to one sided strongly harmonizable moving averages. They are shown to exist when the spectral density with respect to a control measure is made up of the product of meromorphic splittings.

Although some concepts and results are stated for weakly harmonizable processes also, the main emphasis will be for the strongly harmonizable case.

2 SPECTRAL REPRESENTATIONS

In the following work there is always a probability space, (Ω, Σ, P), in the background.

Definition 2.1 For $p \geq 1$ define $\mathbf{L}_0^p(P)$ to be all complex valued $f \in \mathbf{L}^p(P)$ such that $\mathbf{E}(f) = 0$, where $\mathbf{E}(f) \stackrel{\text{def}}{=} \int_\Omega f(\lambda)P(d\lambda)$ is the expectation of f.

Letting $\mathcal{M}_{n,m}$ denote the set of $n \times m$ complex valued matrices, it is frequently advantageous to view $[\mathbf{L}_0^2(P)]^n$ as a left $\mathcal{M}_{n,n}$ module: if $X, Y \in [\mathbf{L}_0^2(P)]^n$ and $A, B \in \mathcal{M}_{n,n}$ then $AX + BY \in [\mathbf{L}_0^2(P)]^n$. Furthermore there exists a $\mathcal{M}_{n,n}$ valued Gramian (see [8]) defined on $[\mathbf{L}_0^2(P)]^n$ given by $[X,Y] \stackrel{\text{def}}{=} \mathbf{E}(XY^*) \in \mathcal{M}_{n,n}$.

We will be considering second order random processes and will only deal with $\mathbf{Z}$ or $\mathbf{R}$ for the index sets. Associated with each of these topological groups is their dual group: $\hat{\mathbf{Z}}$ is the unit circle in $\mathbf{C}$, denoted by $\mathbf{T}$ and $\hat{\mathbf{R}}$ is $\mathbf{R}$. As usual, $\mathbf{T}$ will be thought of as the interval $[-\pi, \pi)$. In order to refer to both the discrete and continuous cases simultaneously, we use the symbol $\mathbf{D}$ to represent the index set, $\mathbf{Z}$ or $\mathbf{R}$, of a process and $\hat{\mathbf{D}}$ its dual. $\mathcal{B}$ will denote the set of Borel subsets for $\hat{\mathbf{D}}$.

Definition 2.2 Let $X_t = (X_t^{(1)}, \ldots, X_t^{(n)})$ be an n-dimensional random process. Define

$$\mathbf{H}_X^-(t) \stackrel{\text{def}}{=} \begin{cases} \overline{\text{sp}}\{X_s^{(j)} : s \le t,\ 1 \le j \le n\} & \text{if } t \in \mathbf{D}, \\ \overline{\text{sp}}\{X_s^{(j)} : s \in \mathbf{D},\ 1 \le j \le n\} & \text{if } t = \infty, \\ \cap_{s\in\mathbf{D}} \mathbf{H}_X^-(s) & \text{if } t = -\infty, \end{cases}$$

where closure is taken in $\mathbf{L}_0^2(P)$. The space $\mathbf{H}_X^-(\infty)$ is referred to the *space of observables* of X_t. Given an n-dimensional vector measure, $Z(\cdot)$, on the space $(\hat{\mathbf{D}}, \mathcal{B})$ to $[\mathbf{L}_0^2(P)]^n$, for every $\Delta \in \mathcal{B}$ we let

$$\mathbf{H}_Z^-(\Delta) \stackrel{\text{def}}{=} \overline{\text{sp}}\{Z^{(j)}(\Delta') : 1 \le j \le n,\ \Delta' \in \mathcal{B} \text{ and } \Delta' \subseteq \Delta\},$$

where closure is again taken in $\mathbf{L}_0^2(P)$. The notation $\text{l.i.m.}_{n\uparrow\infty} Y_n = Y$ is used for the convergence in mean-square, i.e., that $\lim_{n\uparrow\infty} \|Y_n - Y\|_2 = 0$.

Definition 2.3 The complex valued random process, X_t, is *stationary* (stationary in the wide or Khinchine sense) iff (if and only if) the covariance function, $r_X(s,t) \stackrel{\text{def}}{=} \mathbf{E}(X_s X_t^*)$, of X_t is continuous and a function of the difference of its arguments, i.e., if $r_X(s,t) = r_X(s+u, t+u)$ for all $u, s, t \in \mathbf{D}$. Hereafter $\tilde{r}(t) \stackrel{\text{def}}{=} r(0,t)$.

An equivalent definition, by the classical Bochner theorem [18], of a stationary process is one whose covariance function can be represented as

$$\tilde{r}(s) = \int_{\hat{\mathbf{D}}} e^{i\lambda s}\, F(d\lambda),$$

for a unique non-negative bounded measure $F(\cdot)$ on $(\hat{\mathbf{D}}, \mathcal{B})$.

Definition 2.4 A random process, X_t, taking values in $\mathbf{L}_0^2(P)$ is *weakly harmonizable* iff its covariance function can be expressed as

$$r(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t}\, F(d\lambda, d\lambda'), \tag{2.1}$$

where $F(d\lambda, d\lambda')$ is a positive semi-definite bimeasure on $\mathcal{B} \times \mathcal{B}$, hence of finite Fréchet variation. The above integral is a Morse-Transue integral [1]. A random process, X_t, is *strongly harmonizable* iff the bimeasure $F(d\lambda, d\lambda')$ in (2.1) extends to a complex measure (hence has bounded variation in Vitali's sense) on the Borel σ-algebra of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$. In either case, $F(d\lambda, d\lambda')$ is called the *spectral bimeasure* (or spectral measure when $F(d\lambda, d\lambda')$ is a measure) of X_t.

Definition 2.5 An n-dimensional vector of processes,

$$X_t(\cdot) \stackrel{\text{def}}{=} (X_t^1(\cdot), \ldots, X_t^n(\cdot)),$$

is an n-dimensional harmonizable (stationary) process iff for every $1 \times n$ vector of complex numbers, $\mathbf{w}$, the process $\mathbf{w} \cdot X_t$ is harmonizable (stationary).

A standard calculation reveals that equivalent to the above definition of an n-dimensional harmonizable random process is to require that its covariance function be represented as

$$\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t}\, F(d\lambda, d\lambda')$$

where $F(d\lambda, d\lambda')$ is an $n \times n$ matrix array of bimeasures. Likewise, an equivalent definition for an n-dimensional stationary process is that its covariance function can be represented as $\int_{\hat{\mathbf{D}}} e^{i\lambda(t-s)}\, F(d\lambda)$ where $F(d\lambda)$ is an $n \times n$ matrix array of complex valued measures ($F(\Delta)$ will necessarily be a positive semi-definite matrix for all $\Delta \in \mathcal{B}$).

Definition 2.6 An n-dimensional harmonizable process, X_t, has *rank* m iff its spectral measure takes values in the space of $n \times n$ matrices of rank m together with $\mathbf{0}_n$. If $n = m$ the process is said to have *maximal rank*.

One should note that the rank of the covariance function is different from that of the spectral measure. Rank is not defined for all harmonizable processes (see example 5.9). However, when it is defined it is an upper bound for rank $r(s,t)$.

Definition 2.7 An n-dimensional vector measure, $Z(\cdot)$, has *orthogonal increments* (or is said to be *orthogonally scattered*) iff $\Delta \cap \Delta' = \emptyset$ implies that $\mathbf{E}(Z(\Delta)Z^*(\Delta')) = \mathbf{0}_n$.[1]

2.1 Spectral Representation

The following theorem was given by A. Kolmogorov [11]:

Theorem 2.8 *An n-dimensional process, X_t, is stationary iff it has a spectral representation $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$, where $Z(\cdot)$ is a vector measure with orthogonal increments.*[2].

The vector measure, $Z(\cdot)$, in (2.2) can be constructed in the stationary case by a method of H. Cramér [3]; or also by A. Blanc-Lapierre and R. Fortet (see [17] or [12, volume 2, page 149]). A similar construction allows us to construct $Z(\cdot)$ for strongly harmonizable processes [13].

Theorem 2.9 *An n-dimensional process, X_t, is weakly harmonizable iff it has a spectral representation*

$$X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda), \tag{2.2}$$

where $Z(d\lambda)$ is a vector measure, not necessarily of orthogonal increments.

[1] Here * denotes the adjoint operator, i.e., the conjugate transpose operator.

[2] The integral is in the sense of Dunford-Schwartz (see [6, chapter IV, section 10]).

In particular for $a, b \in \hat{\mathbf{D}}$, $a < b$ and $\|Z(\{a\})\|_{[\mathbf{L}_0^2(P)]^n} = \|Z(\{b\})\|_{[\mathbf{L}_0^2(P)]^n} = 0$, we can show that

$$Z([a,b)) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2\pi}\sum_{k\in\mathbf{Z}} \frac{e^{-iak}-e^{-ibk}}{-ik} X_k & \text{discrete case,} \\ \text{l.i.m.}_{T\uparrow\infty} \frac{1}{2\pi}\int_{-T}^{T} \frac{e^{-iat}-e^{-ibt}}{-it} X_t \, dt & \text{continuous case} \end{cases} . \tag{2.3}$$

The spectral bimeasure, $F(\cdot,\cdot)$, of an n-dimensional harmonizable random process satisfies $F(A,B) = \mathbf{E}(Z(A)Z^*(B))$. In the stationary case $Z(d\lambda)$ is orthogonally scattered so that the spectral measure is concentrated on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ and can be written as $F(d\lambda)$.

Lemma 2.10 *Given an n-dimensional strongly harmonizable process, $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$, one has $\mathbf{H}_X^-(\infty) = \mathbf{H}_Z^-(\hat{\mathbf{D}})$.*

Proof: The integral representation of X_t implies $\mathbf{H}_X^-(\infty) \subseteq \mathbf{H}_Z^-(\hat{\mathbf{D}})$. That $\mathbf{H}_Z^-(\hat{\mathbf{D}}) \subseteq \mathbf{H}_X^-(\infty)$ follows from (2.3). ∎

Definition 2.11 Given a probability space (Ω, Σ, P) and taking any other probability space (Ω', Σ', P'), one can "enlarge" (Ω, Σ, P) to an *augmented probability space*, $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, by letting $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) \stackrel{\text{def}}{=} (\Omega \times \Omega', \Sigma \times \Sigma', P \otimes P')$[3].

Let (Ω, Σ, P) be a probability space and $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ be an augmentation of that probability space. For each $\tilde{\omega} \in \tilde{\Omega}$ we can write $\tilde{\omega} = (\omega, \omega')$ where $\omega \in \Omega$ and $\omega' \in \Omega'$. Given a random process X_t on (Ω, Σ, P) we can identify X_t with a random process $\tilde{X}_t$ on the augmented probability space by letting $\tilde{X}_t(\tilde{\omega}) \stackrel{\text{def}}{=} X_t(\omega)$. Since the distributions of X_t and $\tilde{X}_t$ are the same, the two random variables are indistinguishable from a probabilistic point of view.

The following theorem, proved by M. M. Rao [14, theorem 6.1], implicitly uses this identification. The proof (which is not given here) involves using the results of theorem 2.9 along with a Grothendieck type inequality.

Theorem 2.12 (Dilation Theorem) *A random process, X_t, is a harmonizable process iff it has a stationary dilation (Y_t, π), i.e., there exists a stationary process, Y_t, on an augmented probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ along with an orthogonal projection, $\pi : \mathbf{L}_0^2(\tilde{P}) \to \mathbf{L}_0^2(P)$, (where $\mathbf{L}_0^2(P)$ is considered embedded in $\mathbf{L}_0^2(\tilde{P})$) such that $X_t = \pi Y_t$. (Clearly, an n-dimensional version also holds.)*

Given the dilation theorem one can immediately obtain theorem 2.9. However, at the moment, no independent proof of theorem 2.12 is known for obtaining in the representation theorem 2.9.

[3]For every $A \in \Sigma$ and $A' \in \Sigma'$, $P \otimes P'$ is defined by $P \otimes P'(A \times A') \stackrel{\text{def}}{=} P(A)P'(A')$.

2.2 Spectral Measures

To characterize spectral measures it is useful to introduce the following concept.

Definition 2.13 Let A be an index set and $r : A \times A \to \mathcal{M}_{n,n}$ a function. The function $r(\cdot, \cdot)$ is a *positive definite kernel* iff given $m \in \mathbf{Z}^+$, any sets, $\{a_j \in \mathbf{C}^n\}_{j=1}^m$ and $\{t_k \in A\}_{k=1}^m$ then

$$\sum_{j=1}^{m} \sum_{k=1}^{m} a_j r(t_j, t_k) a_k^* \geq 0, \tag{2.4}$$

with equality iff $a_j = 0$ for $1 \leq j \leq m$. If there exists $a_1, \ldots, a_m$ not all equal to zero so that equality holds above, then $F(\cdot, \cdot)$ is a *positive semi-definite kernel.*

For a stationary process with covariance function, $r(s,t) = \tilde{r}(t-s)$, (2.4) can be written as

$$\sum_{j=1}^{m} \sum_{k=1}^{m} a_j \tilde{r}(t_k - t_j) a_k^* \geq 0.$$

This is the definition of positive semi-definite function, $\tilde{r}(\cdot)$, (versus positive semi-definite kernel, $r(\cdot, \cdot)$) and it is the latter that Bochner's classical theorem characterizes. For instance, $\phi(\lambda_1, \lambda_2) = e^{\frac{-(\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2)}{2}}$ is a characteristic function (of a bivariate normal distribution) and hence positive semi-definite (by the theorem of S. Bochner), so $r((\lambda_1, \lambda_2), (\lambda_3, \lambda_4)) \stackrel{\text{def}}{=} \phi(\lambda_3 - \lambda_1, \lambda_4 - \lambda_2)$ is a positive semi-definite kernel.

The following theorem is a consequence of the Kolmogorov Existence Theorem (see [20, page 47]).

Theorem 2.14 *If $r(\cdot, \cdot)$ is an $n \times n$ positive semi-definite kernel with index set A, then it is a covariance function, i.e., there exists a probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ and random variables, $X_t \in \left[\mathbf{L}_0^2(\tilde{P})\right]^n$, such that $r(s,t) = \mathbf{E}(X_s X_t^*)$ for every $s, t \in A$.*

The following theorem generalizes the corresponding stationary result due to H. Cramér [3] and A. Kolmogorov.

Theorem 2.15 *An $n \times n$ array of measures (finite Vitali variation), $F(\cdot, \cdot)$, on $(\hat{\mathbf{D}}, \mathcal{B})$ is a spectral measure iff it is a positive semi-definite kernel.*

Proof: ($\Rightarrow$) If $F(\cdot, \cdot)$ is a spectral measure then there exists a vector measure $Z(\cdot)$ such that $F(\Delta, \Delta') = \mathbf{E}(Z(\Delta) Z^*(\Delta'))$. But this means that for any $m \in \mathbf{Z}^+$ and for all $\Delta_1, \ldots, \Delta_m \in \mathcal{B}$ and for all $a_1, \ldots, a_m \in \mathbf{C}^n$,

$$\sum_{j=1}^{m} \sum_{k=1}^{m} a_j F(\Delta_j, \Delta_k) a_k^* = \mathbf{E}\left(\sum_{j=1}^{m} a_j Z(\Delta_j) \left(\sum_{k=1}^{m} a_k Z(\Delta_k)\right)^*\right) \geq 0.$$

(Indeed, this also shows that spectral bimeasure of a weakly harmonizable process is a positive semi-definite kernel.)

($\Leftarrow$) Theorem 2.14 implies there exist random variables $\{Z_\Delta\}_{\Delta\in\mathcal{B}}$ such that $F(\Delta,\Delta') = \mathbf{E}(Z_\Delta Z^*_{\Delta'})$. (We write $Z(\Delta) \stackrel{\text{def}}{=} (Z_1(\Delta),\ldots,Z_n(\Delta))$ for Z_Δ.) If $Z(\cdot)$ is a vector measure, then $F(\cdot,\cdot)$ is the spectral measure of $X_t \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$, so it suffices to show that $Z(\cdot)$ is a vector measure. Let $\{\Delta_j\}_{j=1}^\infty$ be disjoint elements of $\mathcal{B}$ and $1\le k\le n$. We will first demonstrate the finite additivity of $Z_k(\cdot)$ (and hence $Z(\cdot)$)). For any $m\in\mathbf{Z}^+$ and $\Delta'\in\mathcal{B}$,

$$\begin{aligned}\mathbf{E}(Z_k(\cup_{j=1}^m\Delta_j)Z_k^*(\Delta')) &= F_{kk}(\cup_{j=1}^m\Delta_j,\Delta') = \sum_{j=1}^m F_{kk}(\Delta_j,\Delta')\\ &= \sum_{j=1}^m \mathbf{E}(Z_k(\Delta_j)Z_k^*(\Delta'))\\ &= \mathbf{E}\left((\sum_{j=1}^m Z_k(\Delta_j))Z_k^*(\Delta')\right) \qquad (2.5)\end{aligned}$$

since $F(\cdot,\Delta')$ is an $n\times n$ array of scalar measures. Since the above is true for all $\Delta'\in\mathcal{B}$, it follows that $\sum_{j=1}^m Z_k(\Delta_j)$ is the only member of $\overline{\text{sp}}\{Z_k(\Delta):\Delta\in\mathcal{B}\}$ for which (2.5) is true, so

$$Z_k(\cup_{j=1}^m\Delta_j) = \sum_{j=1}^m Z_k(\Delta_j).$$

To prove σ-additivity of $Z(\cdot)$ note that

$$Z(\cup_{j=1}^\infty\Delta_j) - \sum_{j=1}^m Z(\Delta_j) = Z(\cup_{j=m+1}^\infty\Delta_j).$$

Now

$$\mathbf{E}\left(Z(\cup_{j=m+1}^\infty\Delta_j)Z^*(\cup_{k=m+1}^\infty\Delta_k)\right) = F\left(\cup_{j=m+1}^\infty\Delta_j,\cup_{k=m+1}^\infty\Delta_k\right) \to \mathbf{0}_n,$$

as $m\uparrow\infty$, since $F(\cdot,\cdot)$ is a measure and $\cap_{m\in\mathbf{Z}^+}\left[(\cup_{j=m+1}^\infty\Delta_j)\times(\cup_{k=m+1}^\infty\Delta_k)\right] = \emptyset$. Thus

$$\operatorname*{l.i.m.}_{m\uparrow\infty} Z(\cup_{j=m+1}^\infty\Delta_j) = 0 \qquad\Longrightarrow\qquad \sum_{j=1}^\infty Z(\Delta_j) = Z(\cup_{j=1}^\infty\Delta_j). \qquad \blacksquare$$

The above theorem shows that if the measure on $\hat{\mathbf{D}}\times\hat{\mathbf{D}}$ is a positive semi-definite kernel, then it is a spectral measure for an n-dimensional harmonizable process, $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$. Thus for any spectral measure $F(\cdot,\cdot)$,

$$\begin{aligned}|F_{jk}(\Delta,\Delta')|^2 &= |\mathbf{E}(Z_j(\Delta)Z_k^*(\Delta'))|^2\\ &\le |\mathbf{E}(Z_j(\Delta)Z_j^*(\Delta))||\mathbf{E}(Z_k(\Delta')Z_k^*(\Delta'))|\\ &= |F_{jj}(\Delta,\Delta)||F_{kk}(\Delta',\Delta')|.\end{aligned}$$

Proposition 2.16 *If $f(\cdot,\cdot)$ is a measurable positive semi-definite kernel on $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ taking on values in the space of $n \times n$ complex matrices and $\mu(\cdot)$ is a measure on $\hat{\mathbf{D}}$ such that*

$$\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} f_{jk}(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda') < \infty \qquad 1 \le j,k \le n$$

then $f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')$ is a positive semi-definite kernel (and hence a spectral measure by theorem 2.15).

Proof: It will be shown that for any $m \in \mathbf{Z}^+$, any subset $\{\Delta_j\}_{j=1}^m$ of $\mathcal{B}$ (the Borel sets of $\hat{D}$) and any set, $\{a_j\}_{j=1}^m \subset \mathbf{C}^n$, the following holds:

$$\sum_{j=1}^m \sum_{k=1}^m a_j \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')a_k^* \ge 0. \tag{2.6}$$

If the support of $\mu(\cdot)$ consists of a finite number of points $b_1,\ldots,b_K$ so that $\mu(\cdot) \stackrel{\text{def}}{=} \sum_{j=1}^K c_j\chi_{\{b_j\}}(\cdot)$, then (2.6) can be established from

$$\begin{aligned}
&\sum_{j=1}^m \sum_{k=1}^m a_j \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')a_k^* \\
&\quad= \sum_{j=1}^m \sum_{k=1}^m a_j \sum_{j'=1}^K \sum_{k'=1}^K c_{j'}\chi_{\Delta_j}(b_{j'})f(b_{j'},b_{k'})c_{k'}\chi_{\Delta_k}(b_{k'})a_k^* \\
&\quad= \sum_{j=1}^m \sum_{j'=1}^K \sum_{k=1}^m \sum_{k'=1}^K \left[a_j c_{j'}\chi_{\Delta_j}(b_{j'})\right] f(b_{j'},b_{k'}) \left[a_k c_{k'}\chi_{\Delta_k}(b_{k'})\right]^* \ge 0,
\end{aligned}$$

since $f(\cdot,\cdot)$ is a positive semi-definite kernel.

Even if the support of $\mu(\cdot)$ does not consist of a finite number of points, there exists a sequence of simple functions,

$$\tilde{s}_\ell(\lambda,\lambda') \stackrel{\text{def}}{=} \sum_{j=1}^{m_\ell} \sum_{k=1}^{m_\ell} c_{\ell,j,k}\chi_{\Delta_{\ell,j}}(\lambda)\chi_{\Delta_{\ell,k}}(\lambda')$$

such that

1. if $1 \le j,k \le m$ and $(\lambda,\lambda') \in \Delta_{\ell,j}\times\Delta_{\ell,k}$ then $|\tilde{s}_\ell(\lambda,\lambda') - f(\lambda,\lambda')| < \frac{1}{\ell}$,
2. $\Delta_{\ell,j} \cap \Delta_{\ell,k} = \emptyset$ iff $j \ne k$, and
3. $\iint_{\Delta_j\times\Delta_k} \tilde{s}_\ell(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda') \to \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')$ for $1 \le j,k \le m$.

Choose $\lambda_{\ell,j} \in \Delta_{\ell,j}$ for $1 \le j \le m_\ell$ and define a sequence of simple functions,

$$s_\ell(\lambda,\lambda') \stackrel{\text{def}}{=} \sum_{j=1}^{m_\ell}\sum_{k=1}^{m_\ell} f(\lambda_{\ell,j},\lambda_{\ell,k})\chi_{\Delta_{\ell,j}}(\lambda)\chi_{\Delta_{\ell,k}}(\lambda').$$

Using the fact that $f(\cdot,\cdot)$ is a positive semi-definite kernel, a simple calculation shows that $s_\ell(\cdot,\cdot)$ is also a positive semi-definite kernel. Since

$$\|\tilde{s}_\ell - s_\ell\|_\infty \le \frac{1}{\ell}|\mu|^2(\hat{\mathbf{D}}) \to 0,$$

it follows that

$$\iint_{\Delta_j\times\Delta_k} s_\ell(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda') \to \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')$$

for $1 \le j,k \le m$.

Define $\mu_\ell(\cdot)$ to be a scalar measure with support on $\{\lambda_{\ell,j}\}_{j=1}^{m_\ell}$ such that $\mu_\ell(\{\lambda_{\ell,j}\}) \stackrel{\text{def}}{=} \mu(\Delta_{\ell,j})$. Since $\mu_\ell(\cdot)$ has support on a finite set and $s_\ell(\cdot,\cdot)$ is a positive semi-definite kernel, the argument given at the beginning of the proof shows $s_\ell(\lambda,\lambda')\,\mu_\ell(d\lambda)\bar{\mu}_\ell(d\lambda')$ is a positive semi-definite kernel on the set $R(\mathcal{B}\times\mathcal{B})$ of all rectangles in $\mathcal{B}\times\mathcal{B}$.

Since $s_\ell(\lambda,\lambda')\,\mu_\ell(d\lambda)\bar{\mu}_\ell(d\lambda')$ is a positive semi-definite kernel on $R(\mathcal{B}\times\mathcal{B})$,

$$\begin{aligned}
0 &\le \sum_{j=1}^{m}\sum_{k=1}^{m} a_j \iint_{\Delta_j\times\Delta_k} s_\ell(\lambda,\lambda')\,\mu_\ell(d\lambda)\bar{\mu}_\ell(d\lambda')a_k^* \\
&= \sum_{j=1}^{m}\sum_{k=1}^{m} a_j \iint_{\Delta_j\times\Delta_k} s_\ell(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')a_k^* \\
&\to \sum_{j=1}^{m}\sum_{k=1}^{m} a_j \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda)\bar{\mu}(d\lambda')a_k^*
\end{aligned}$$

which establishes (2.6) where $\mu(\cdot)$ has arbitrary support. ∎

Proposition 2.16 need not hold if the measure does not split as seen from the following example.

Example 2.17 Let $a,b \in \hat{\mathbf{D}}$ with $a \neq b$. Define

$$f(\lambda,\lambda') = \begin{cases} 1 & \text{if } (\lambda,\lambda') = (a,a) \text{ or } (\lambda,\lambda') = (b,b), \\ -\frac{1}{2} & \text{if } (\lambda,\lambda') = (a,b) \text{ or } (\lambda,\lambda') = (b,a), \\ 0 & \text{otherwise} \end{cases}$$

and define $\mu(\cdot,\cdot)$ with support on $\{(a,b),(b,a)\}$ with

$$\mu(\{(a,b)\}) = \mu(\{(b,a)\}) = 1.$$

Let $\Delta_1 = \{a\}$ and $\Delta_2 = \{b\}$. Furthermore let $a_1 = a_2 = 1$ and observe that

$$\sum_{j=1}^{2}\sum_{k=1}^{2} a_j \iint_{\Delta_j\times\Delta_k} f(\lambda,\lambda')\,\mu(d\lambda,d\lambda')a_k^* = -1.$$

Note that $f(\cdot,\cdot)$ is a positive semi-definite kernel. However $f(\lambda,\lambda')\,\mu(d\lambda,d\lambda')$ is not a positive semi-definite kernel and hence by theorem 2.15 is not a spectral measure.

3 FREQUENCY AND TIME DOMAINS

Definition 3.1 Given an n-dimensional strongly harmonizable process, X_t, a complex Borel measure, $\mu(\cdot,\cdot)$, is a *controlling measure* for X_t iff the spectral measure, $F(\cdot,\cdot)$, of X_t is absolutely continuous with respect to it. If in addition, the controlling measure is non-negative it is said to be a *control measure* (See [10, page 64]).

A controlling measure always exists. For instance, let

$$\mu(d\lambda,d\lambda') = \sum_{1\le j,k\le n} |F_{jk}|(d\lambda,d\lambda') \ge 0.$$

This control measure is dominated by all other controlling measures in the sense that if $\tilde{\mu}$ is also a control measure, then $F_{ij} \ll \mu \ll \tilde{\mu}$.

Definition 3.2 Let X_t be an n-dimensional harmonizable random process with spectral representation $X_t = \int_{\hat{D}} e^{it\lambda}\,Z(d\lambda)$. For $p \in \mathbf{Z}^+$ define an equivalence relation on the set of $p\times n$ matrix valued functions by $A(\cdot) \sim B(\cdot)$ iff $\|\int_{\hat{\mathbf{D}}}(A-B)(\lambda)\,Z(d\lambda)\|_{[\mathbf{L}_0^2(P)]^p} = 0$. Let $\mathbf{L}^2(F,p)$, the *spectral domain* of X_t, be the set of equivalent classes $[A(\cdot)]$ such that $\int_{\hat{\mathbf{D}}} A(\lambda)\,Z(d\lambda) \in [\mathbf{L}_0^2(P)]^p$.

If $F(\cdot,\cdot)$ is the spectral measure of X_t and $A(\cdot)$ and $B(\cdot)$ are $p\times n$ matrix functions, then $A \sim B$ iff

$$\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} (A-B)(\lambda)F(d\lambda,d\lambda')(A-B)^*(\lambda') = \mathbf{0}_p,$$

where the integrals are defined componentwise. In fact the spectral domain of X_t depends only on the spectral measure $F(\cdot,\cdot)$ of X_t and not on X_t itself since $Y_A \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} A(\lambda)\,Z(d\lambda) \in [\mathbf{L}_0^2(P)]^p$ iff

$$\mathbf{E}(Y_A Y_A^*) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} A(\lambda)F(d\lambda,d\lambda')A^*(\lambda')$$

exists.

The theorem below says that given an n-dimensional strongly harmonizable process, X_t, the Hilbert space of the observables, $\mathbf{H}_X^-(\infty)$, is isomorphic to the spectral domain, $\mathbf{L}^2(F,1)$, of X_t. Since the spectral domain consists of $1 \times n$ arrays of scalar functions, the latter is easier to study than the space of random variables, $\mathbf{H}_X^-(\infty)$. The theorem tells us that the study of the spectral domain is the study of the space of observables. This theorem in the weakly harmonizable case with positive definite bimeasures was given in D. Chang and M. M. Rao [1, theorem 9.4]. M. M. Rao in [15], using methods distinct from those used here, obtained the theorem for the case that $F(\cdot,\cdot)$ is a positive definite kernel (versus a positive semi-definite kernel). Using different proofs, both M. Rosenberg [16, §5] and Y. Rozanov [18, pages 30-32] separately proved the stationary version of the theorem below. We will use the stationary version in proving the strongly harmonizable version, employing the dilation theorem.

Theorem 3.3 *Given an n-dimensional strongly harmonizable random process, X_t, the vector space $\mathbf{L}^2(F_X,p)$ along with the inner product*

$$((A,B)) \stackrel{\text{def}}{=} \operatorname{tr}\left(\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} A(\lambda)F_X(d\lambda,d\lambda')B^*(\lambda')\right)$$

is a Hilbert space. The map $\Psi : \mathbf{L}^2(F_X,p) \to \left[\mathbf{H}_X^-(\infty)\right]^p$ defined by

$$\Psi : A \mapsto X_A \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} A(\lambda)\, Z(d\lambda)$$

is an isometric isomorphism between the two Hilbert spaces.

Proof: It is clear that $\mathbf{L}^2(F_X,p)$ is a vector space and $((\cdot,\cdot))$ is an inner product.

Let $w_{jk}(t)$ be defined to be the $p \times n$ matrix function in λ such that the jk entry is $e^{it\lambda}$ and all other entries are equal to the zero function. Define

$$S \stackrel{\text{def}}{=} \operatorname{sp}\{w_{jk}(\cdot) : 1 \le j \le p, 1 \le k \le n\} \subseteq \mathbf{L}^2(F_X,p)$$

and

$$V \stackrel{\text{def}}{=} \left[\operatorname{sp}\{X_t^{(j)} : t \in \mathbf{D}, 1 \le j \le n\}\right]^p \subseteq \left[\mathbf{H}_X^-(\infty)\right]^p.$$

We now define a norm preserving isomorphism, $\tilde{\Psi} : S \to V$ by

$$\tilde{\Psi} : w_{jk}(\cdot) \mapsto \int_{\hat{\mathbf{D}}} w_{jk}(\lambda)\, Z_X(d\lambda) = (0,\ldots,0,X_t^{(k)},0,\ldots,0), \tag{3.7}$$

where $(0,\ldots,0,X_t^{(k)},0,\ldots,0)$ is the p-dimensional random process that is zero everywhere except for its j-th coordinate, where it is equal to $X_t^{(k)}$. Notice that if $\theta \in S$ then

$$((\theta,\theta)) = \operatorname{tr}\left(\mathbf{E}(\tilde{\Psi}(\theta)\tilde{\Psi}^*(\theta))\right) = \sum_{j=1}^{p} \|\tilde{\Psi}^{(j)}(\theta)\|_2^2 = \|\tilde{\Psi}(\theta)\|^2_{[\mathbf{L}_0^2(P)]^p}.$$

We will show that

1. if $\Phi_m \in S$ is a Cauchy sequence, then there exists a $\Phi \in \mathbf{L}^2(F_X, p)$ such that $\Phi_m \to \Phi$ in the $((\cdot,\cdot))$ inner product norm.

2. S is dense in $\mathbf{L}^2(F_X, p)$.

These two claims will show immediately that $\mathbf{L}^2(F_X, p)$ is a Hilbert space. Letting $\Phi \in \mathbf{L}^2(F_X, p)$, claim (2) tells us that there exist $\Phi_m \in S$ such that $\Phi_m \to \Phi$. We can now define $\Psi(\Phi)$ by $\Psi : \Phi \mapsto \text{l.i.m.}_{m\uparrow\infty} \tilde{\Psi}(\Phi_m)$. This is well defined since $\tilde{\Psi}(\cdot)$ is an isometry onto its image and if $\Phi'_m \in S$ also has the property that $\Phi'_m \to \Phi$, then

$$\lim_{m\uparrow\infty} \|\tilde{\Psi}(\Phi_m) - \tilde{\Psi}(\Phi'_m)\|_{[\mathbf{L}_0^2(P)]^p} = \lim_{m\uparrow\infty} \|\Phi_m - \Phi'_m\|_{\mathbf{L}^2(F_X,p)} = 0.$$

Since $\text{sp}\{X_t^{(j)} : t \in \mathbf{D}, 1 \le j \le n\}$ is dense in $\mathbf{H}_X^-(\infty)$, equation (3.7) implies that $\Psi(\cdot)$ is an isomorphism between $\mathbf{L}^2(F_X, p)$ and $\left[\mathbf{H}_X^-(\infty)\right]^p$ and is also an isometry since $\tilde{\Psi}(\cdot)$ is. Thus we are left with proving the two claims above.

To prove claim (1) note that $\{\tilde{\Psi}(\Phi_m)\}$ is a Cauchy sequence in $\left[\mathbf{H}_X^-(\infty)\right]^p$ and since $\mathbf{H}_X^-(\infty)$ is complete, there exists a $W \in \left[\mathbf{H}_X^-(\infty)\right]^p$ such that $\text{l.i.m.}_{m\uparrow\infty} \tilde{\Psi}(\Phi_m) = W$.

By the dilation theorem (theorem 2.12) there exists a stationary dilation, (Y_t, π) such that $X_t = \pi Y_t$.

Since $\pi(\left[\mathbf{H}_Y^-(\infty)\right]^p) = \left[\mathbf{H}_X^-(\infty)\right]^p$ there exists $\tilde{W} \in \left[\mathbf{H}_Y^-(\infty)\right]^p$ such that $\pi\tilde{W} = W$. The stationary version of the above theorem states that there exist $\Phi \in \mathbf{L}^2(F_Y, p)$ such that $\tilde{W} = Y_\Phi \overset{\text{def}}{=} \int_{\hat{\mathbf{D}}} \Phi(\lambda)\, Z_Y(d\lambda)$. Thus

$$W = \pi(Y_\Phi) = \int_{\hat{\mathbf{D}}} \Phi(\lambda)\, \pi(Z_Y(d\lambda)) = \int_{\hat{\mathbf{D}}} \Phi(\lambda)\, Z_X(d\lambda) = X_\Phi.$$

Since π is a bounded operator, one can move π through the integral (see [6, IV.10.8f]). Thus

$$\begin{aligned} \|\Phi - \Phi_m\|^2_{((\cdot,\cdot))} &= \|\int_{\hat{\mathbf{D}}} \Phi(\lambda)\, Z_X(d\lambda) - \int_{\hat{\mathbf{D}}} \Phi_m\, Z_X(d\lambda)\|^2_{\left[\mathbf{L}_0^2(P)\right]^p} \\ &= \|W - \tilde{\Psi}(\Phi_m)\|^2_{\left[\mathbf{L}_0^2(P)\right]^p} \to 0 \qquad \text{as } m \uparrow \infty, \end{aligned}$$

(since $\text{l.i.m.}_{m\uparrow\infty} \tilde{\Psi}(\Phi_m) = W$) so claim (1) is proven.

Claim (2) follows from the fact that trigonometric polynomials are dense in $\mathbf{L}^2(\mu)$ for any finite Radon measure $\mu(\cdot,\cdot)$[4]. ∎

The weakly harmonizable version of the above theorem is used in the following definition.

[4]The theorem will also hold for the weakly harmonizable case if we can show that trigonometric polynomials are dense in $\mathbf{L}^2(\mu)$ for any finite Radon bimeasure $\mu(\cdot,\cdot)$.

Definition 3.4 Given an n-dimensional weakly harmonizable process, X_t, and $Y \in \left[\mathbf{H}_X^-(\infty)\right]^p$, the *spectral characteristic* of Y with respect to X_t is (using the notation of theorem 3.3) $\Psi^{-1}(Y) \in \mathbf{L}^2(F_X, p)$.

4 SPLITTING PROCESSES

In this section we analyze a subclass of n-dimensional processes.

Definition 4.1 An n-dimensional random process is called *splitting* (or *a splitting process*) iff its covariance function factors as $r(s,t) = G(s)G^*(t)$ where $G(\cdot)$ is an $n \times q$ matrix valued function on $\mathbf{D}$.

Theorem 4.2 *An n-dimensional random process, X_t, is a splitting process iff* $\dim \mathbf{H}_X^-(\infty) = p < \infty$. *Furthermore, X_t can be represented as $X_t = H(t)W$ where $H(\cdot)$ is a function on $\mathbf{D}$ taking its values in the space of $n \times p$ matrices and W is a p-dimensional random variable such that $\mathbf{E}(WW^*) = \mathbf{I}_p$.*

Proof: ($\Rightarrow$) Assume X_t is splitting, i.e., its covariance function $r_X(s,t) = G(s)G^*(t)$ for some $n \times q$ matrix-valued function $G(\cdot)$. Let V be a q-dimensional random vector such that $\mathbf{E}(VV^*) = \mathbf{I}_q$. Define $Y_t \stackrel{\text{def}}{=} G(t)V$ and notice that $\dim \mathbf{H}_Y^-(\infty) \leq q < \infty$. Since $r_Y(\cdot,\cdot) = r_X(\cdot,\cdot)$ there exists an Hilbert space isomorphism $\Phi : \mathbf{H}_X^-(\infty) \to \mathbf{H}_Y^-(\infty)$ such that $\Phi(X_t) = Y_t$ for every $t \in \mathbf{D}$. Thus $\dim \mathbf{H}_X^-(\infty) < \infty$.

($\Leftarrow$) Assume $\dim \mathbf{H}_X^-(\infty) = p < \infty$. Let W be a p-dimensional random vector consisting of an orthonormal basis for $\mathbf{H}_X^-(\infty)$. Thus $\mathbf{E}(WW^*) = \mathbf{I}_p$. For each $t \in \mathbf{D}$ let $H(t)$ be an $n \times p$ matrix such that $X_t = H(t)W$. It is clear that X_t is splitting since $\mathbf{E}(X_s X_t^*) = H(s)\mathbf{E}(WW^*)H^*(t) = H(s)H^*(t)$. ∎

Corollary 4.3 *Finite linear combinations of splitting processes are splitting processes.*

Proof: Let $Y_t = \sum_{j=1}^m a_j X_t^j$, where X_t^j are splitting processes. By theorem 4.2, $p_j = \dim \mathbf{H}_{X^j}^-(\infty) < \infty$. Thus $\dim \mathbf{H}_Y^-(\infty) \leq \sum_{j=1}^m p_j < \infty$. Using theorem 4.2 again, the corollary is proven. ∎

Note that the representation $X_t = H(t)W$ is not unique since there does not exist a unique orthonormal basis for $\mathbf{H}_X^-(\infty)$. Suppose $\dim \mathbf{H}_X^-(\infty) = p < \infty$ and $X_t = H_1(t)W_1 = H_2(t)W_2$ where $\mathbf{E}(W_1W_1^*) = \mathbf{E}(W_2W_2^*) = \mathbf{I}_p$. It is not difficult to see that there exists a unique $p \times p$ unitary matrix, U, such that $W_1 = UW_2$ and $H_2(t) = H_1(t)U$ for all $t \in \mathbf{D}$.

4.1 Characterization of Splitting Harmonizable Processes

Definition 4.4 A process is *clipped* on a set $\Delta \subseteq \mathbf{D}$ iff $X_t = 0$ for $t \notin \Delta$.

Theorem 4.5 *If an n-dimensional discrete process, X_t, is clipped on a compact set then it is a splitting harmonizable process with spectral measure absolutely continuous with respect to planar Lebesgue measure.*

Proof: Theorem 4.2 implies that X_t is a splitting process and it can be represented as $X_t = H(t)W$ where $\mathbf{E}(WW^*) = \mathbf{I}_p$ and $H(\cdot)$ is an $n \times p$ matrix valued function. Since X_t is clipped on a compact set there exists an N such that $H(t) = \mathbf{0}_{n,p}$ for $|t| \geq N$. For $m \geq 0$ define partial sums, $s_m(t) \stackrel{\text{def}}{=} \sum_{|j| \leq m} H(j)e^{ijt}$, and their corresponding Cesaro sums, $\sigma_m(t) \stackrel{\text{def}}{=} \frac{1}{m}\left(s_0(t) + \ldots + s_{m-1}(t)\right)$. If $m > N$ then $\sigma_m(t) = \sum_{j=-N}^{N} \frac{m-j}{m} H(j)e^{ijt}$. One now sees that for $m > N$, $1 \leq \ell \leq n$ and $1 \leq k \leq p$,

$$
\begin{aligned}
\int_{\mathbf{T}} |\,(\sigma_m(t))_{\ell k}\,|\,dt &\leq \int_{\mathbf{T}} \sum_{j=-N}^{N} |\frac{m-j}{m}\,(H(j))_{\ell k}\,e^{ijt}|\,dt \\
&\leq \int_{\mathbf{T}} \sum_{j=-N}^{N} |\,(H(j))_{\ell k}\,|\,dt \\
&= \sum_{j=-N}^{N} |\,(H(j))_{\ell k}\,| = \sum_{j \in \mathbf{Z}} |\,(H(j))_{\ell k}\,| < \infty.
\end{aligned}
$$

Since the Cesaro means of $\sum_{j \in \mathbf{Z}} H(j)e^{ijt}$ are bounded in the $\mathbf{L}^1(dt)$ norm, $H(\cdot)$ is the Fourier transform of an $n \times p$ matrix of finite measures, $G(d\lambda)$ (see [7, page 23]). Furthermore, since for $1 \leq \ell \leq n$ and $1 \leq k \leq p$ one has $\lim_{m \uparrow \infty} \frac{1}{2m+1} \sum_{|j| \leq m} |\,(H(j))_{\ell k}\,|^2 = 0$, a result of N. Wiener [9, page 42] says $G(d\lambda)$ is absolutely continuous with respect to Lebesgue measure. Thus there exists an $n \times p$ matrix valued function, $P(\cdot)$, such that $G(d\lambda) = P(\lambda)d\lambda$. Continuing,

$$
\begin{aligned}
(r_X(s,t))_{jk} &= \sum_{\ell=1}^{p} H_{j\ell}(s)\,(H^*(t))_{\ell k} \\
&= \sum_{\ell=1}^{p} \left(\int_{\mathbf{T}} e^{-is\lambda}\,G(d\lambda)\right)_{j\ell} \left(\left(\int_{\mathbf{T}} e^{-it\lambda'}\,G(d\lambda')\right)^*\right)_{\ell k} \\
&= \sum_{\ell=1}^{p} \left(\int_{\mathbf{T}} e^{-is\lambda} P(\lambda)_{j\ell}\,d\lambda\right) \left(\int_{\mathbf{T}} e^{it\lambda'} P^*(\lambda')_{\ell k}\,d\lambda'\right) \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{-is\lambda+it\lambda'} \sum_{\ell=1}^{p} P(\lambda)_{j\ell} P^*(\lambda')_{\ell k}\,d\lambda\,d\lambda' \\
&= \left(\iint_{\mathbf{T}\times\mathbf{T}} e^{-is\lambda+it\lambda'} P(\lambda)P^*(\lambda')\,d\lambda\,d\lambda'\right)_{jk}.
\end{aligned}
$$

Letting $\tilde{P}(\lambda) \stackrel{\text{def}}{=} P(-\lambda)$ we have

$$r_X(s,t) = \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'}\,\tilde{P}(\lambda)\tilde{P}^*(\lambda')\,d\lambda\,d\lambda'. \qquad \blacksquare$$

Not all strongly harmonizable splitting process are clipped on compact sets. The following result of D. Chang and M. M. Rao [2, Proposition 1] however still holds.

Theorem 4.6 (Characterization of Split Harmonizable Processes) *An n-dimensional random process, X_t, is a splitting (weakly or strongly) harmonizable process iff*

$$r_X(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'}\,H(d\lambda)H^*(d\lambda') = \hat{H}(-s)\hat{H}^*(-t), \tag{4.8}$$

where $\hat{H}(\cdot)$ is a Fourier transform of an $n \times p$ matrix of Borel measures on $\hat{\mathbf{D}}$. Thus X_t is a strongly harmonizable process. (See also example 4.8 below.)

Proof: ($\Rightarrow$) We will consider the case when X_t is a continuous parameter splitting harmonizable process (the discrete case is similar). Using an inversion formula for covariances [14, theorem 8.2] (which is easily seen to be valid in the multivariate case) we have

$$F(\Delta,\Delta') = \lim_{T_1,T_2\uparrow\infty}\int_{-T_1}^{T_1}\int_{-T_2}^{T_2}\left[\frac{e^{-is\lambda_1} - e^{-is\lambda_2}}{-is}\right]\left[\frac{e^{it\lambda_2'} - e^{-it\lambda_1'}}{it}\right] r(s,t)\,ds\,dt,$$

where $\Delta \stackrel{\text{def}}{=} (\lambda_1,\lambda_2)$ and $\Delta' \stackrel{\text{def}}{=} (\lambda_1',\lambda_2')$ are intervals of $\mathbf{R}$ chosen such that $F(\{\lambda_1\},\{\lambda_1'\}) = 0 = F(\{\lambda_2\},\{\lambda_2'\})$. Substituting $r(s,t) = G(s)G^*(t)$ into the above equation gives us $F(\Delta,\Delta') = H(\Delta)H^*(\Delta')$ for

$$H(\Delta) \stackrel{\text{def}}{=} \lim_{T\uparrow\infty}\int_{-T}^{T}\frac{e^{-is\lambda_1} - e^{-is\lambda_2}}{-is}G(s)\,ds.$$

Since $F(\cdot,\cdot)$ is a bimeasure, $H(\cdot)$ is σ-additive on the class of all such intervals which are also continuity intervals of $H(\cdot)$. The standard theory says that $H(\cdot)$ has a unique extension to a matrix valued measure on the Borel sets of $\mathbf{R}$. Thus $F(d\lambda,d\lambda') = H(d\lambda)H^*(d\lambda')$ and $r(s,t) = \iint_{\mathbf{R}\times\mathbf{R}} e^{is\lambda-it\lambda'}\,H(d\lambda)H^*(d\lambda')$, from which (4.8) follows.

If X_t is a discrete parameter splitting harmonizable random variable, the same proof as above is applicable, though it is a little simpler since one does not need to take limits for the inversion formula on $\mathbf{T}$.

($\Leftarrow$) Obvious. $\blacksquare$

The theorem shows that if a process is splitting and weakly harmonizable then it is strongly harmonizable. Thus harmonizable splitting is not ambiguous. One can even say more:

Corollary 4.7 *Given an n-dimensional weakly harmonizable process, X_t, and a stationary dilation, (Y_t, π), then X_t is strongly harmonizable if $\mathbf{H}_X^-(\infty)$ is finite dimensional or has finite codimension in $\mathbf{H}_Y^-(\infty)$.*

Proof: If $\mathbf{H}_X^-(\infty)$ has finite dimension then it is a splitting process by theorem 4.2. The above theorem states that any weakly harmonizable process that splits must be strongly harmonizable.

If $\mathbf{H}_X^-(\infty)$ has finite codimension with respect to $\mathbf{H}_Y^-(\infty)$, then $X_t - Y_t$ has a finite dimensional set of observables and is weakly harmonizable so by the first part of the corollary, $X_t - Y_t$ is strongly harmonizable. Thus $X_t = Y_t + (X_t - Y_t)$ is strongly harmonizable since Y_t and $X_t - Y_t$ are Gramian orthogonal (see remarks following definition 2.1). ∎

However, not all splitting processes are even weakly harmonizable as the following example shows.

Example 4.8 Let $H(\cdot)$ be a scalar function on $\mathbf{D}$ which is not a Fourier transform of a measure. An unbounded function will do. Then $r(s,t) = H(-s)H^*(-t)$ is a positive semi-definite kernel and hence, by theorem 2.14, there exists a process, X_t, with the above covariance function. If X_t is splitting harmonizable there would exist, $\hat{G}(\cdot)$, the Fourier transform of a measure such that $\hat{G}(s)\hat{G}^*(t) = r(-s,-t) = H(s)H^*(t)$. Fixing a t such that $\hat{G}(t) \neq 0$ we have

$$\hat{G}(s) = \overline{\left(\frac{H(t)}{\hat{G}(t)}\right)} H(s).$$

Thus $\hat{G}(\cdot)$ is a constant multiple of $H(\cdot)$ which would imply that $H(t)$ is a Fourier transform of a measure, which it is not. Thus the assumption that X_t was a splitting harmonizable process was false.

We are led to study a particular subset of splitting process.

Definition 4.9 An n-dimensional random process, X_t, is a *splitting process of full splitting rank m* iff there exists a representation of the covariance function, $r_X(s,t) = G(s)G^*(t)$, where $G(\cdot)$ takes values in the space of $n \times m$ matrices of rank m.

The splitting rank of a splitting process may not exist so is not defined for all splitting processes. A fact pertaining to the above definition is given in the following lemma.

Lemma 4.10 *Given an n-dimensional random variable, X, then the rank of $\mathbf{E}(XX^*)$ is the same as the dimension of X, i.e., the number of its linearly independent components.*

The following is a characterization of splitting harmonizable processes with full splitting rank.

Theorem 4.11 (Splitting Harmonizable with Full Splitting Rank) *Let X_t be an n-dimensional process. Then the following are equivalent:*

1. *X_t is a splitting harmonizable process with full splitting rank representation $r(s,t) = \hat{H}(-s)\hat{H}^*(-t)$ where $H(\cdot)$ is an $n \times m$ matrix of Borel measures.*

2. *$X_t = \hat{H}(-t)W$ where $\mathbf{E}(WW^*) = \mathbf{I}_m$ and $H(\cdot)$ is an $n \times m$ matrix of Borel measures and $\hat{H}(\cdot)$ is of rank m.*

3. *$X_t = \int_{\mathbf{T}} e^{it\lambda}\, Z(d\lambda)$ where $Z(\cdot) = H(\cdot)W$, where $\hat{H}(\cdot)$ has rank m, and where $\mathbf{E}(WW^*) = \mathbf{I}_m$.*

Proof: (1 $\Rightarrow$ 2) Define $\Psi(\cdot)$ such that $\Psi(s)\hat{H}(-s) = \mathbf{I}_m$ ($\Psi(\cdot)$ need not be unique). Define $W_t \stackrel{\text{def}}{=} \Psi(t)X_t$. Then for all $s,t \in \mathbf{D}$,

$$\begin{aligned}
\mathbf{E}(W_sW_t^*) &= \mathbf{E}(\Psi(s)X_sX_t^*\Psi^*(t)) \\
&= \Psi(s)\mathbf{E}(X_sX_t^*)\Psi^*(t) = \Psi(s)\hat{H}(-s)\hat{H}^*(-t)\Psi^*(t) = \mathbf{I}_m.
\end{aligned}$$

Thus $\mathbf{E}((W_s - W_t)(W_s - W_t)^*) = \mathbf{0}_m$ which implies that $W_s = W_t$ a.e. for all s and t, so we can write W for W_t. It is also clear that each component of W is a member of $\mathbf{L}_0^2(P)$ since it is a linear combination of the components of X_t.

It remains to show that $X_t = \hat{H}(-t)W$, a.e.. Using the fact that $W = \Psi(t)X_t$ for every $t \in \mathbf{D}$, the result follows from:

$$\begin{aligned}
&\mathbf{E}((\hat{H}(-t)W - X_t)(\hat{H}(-t)W - X_t)^*) \\
&= \mathbf{E}(\hat{H}(-t)WW^*\hat{H}^*(-t)) - \mathbf{E}(\hat{H}(-t)WX_t^*) \\
&\qquad -\mathbf{E}(X_tW^*\hat{H}^*(-t)) + \mathbf{E}(X_tX_t^*) \\
&= r(t,t) - \hat{H}(-t)\Psi(t)r(t,t) - r(t,t)\Psi^*(t)\hat{H}^*(-t) + r(t,t) \\
&= r(t,t) - \hat{H}(-t)\Psi(t)\hat{H}(-t)\hat{H}^*(-t) - \hat{H}(-t)\hat{H}^*(-t)\Psi^*(t)\hat{H}^*(-t) + r(t,t) \\
&= r(t,t) - r(t,t) - r(t,t) + r(t,t) = 0.
\end{aligned}$$

(2 $\Rightarrow$ 3) Observe that

$$\int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda) = X_t = \hat{H}(-t)W = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, H(d\lambda)W.$$

Looking at the first and last expressions in the above equations and using the fact that Fourier transforms of vector measures are unique we have the result.

(3 ⇒ 1) Note that

$$\begin{aligned} r(s,t) &= \mathbf{E}(X_s X_t^*) = \mathbf{E}\left(\left(\int_{\hat{\mathbf{D}}} e^{is\lambda}\, Z(d\lambda)\right)\left(\int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)\right)^*\right) \\ &= \mathbf{E}\left(\left(\int_{\hat{\mathbf{D}}} e^{is\lambda}\, H(d\lambda)W\right)\left(\int_{\hat{\mathbf{D}}} e^{it\lambda}\, H(d\lambda)W\right)^*\right) \\ &= \hat{H}(-s)\mathbf{E}(WW^*)\hat{H}(-t)^* = \hat{H}(-s)\hat{H}^*(-t). \qquad \blacksquare \end{aligned}$$

From the above proof we note that if X_t is an n-dimensional process (perhaps non-harmonizable), then X_t is a splitting process with full splitting rank covariance representation, $r_X(s,t) = G(s)G^*(t)$, iff $X_t = G(t)W$ where $\mathbf{E}(WW^*) = \mathbf{I}_m$ and $G(\cdot)$ is of full rank.

If $Z(d\lambda)$ is a vector measure consisting of a single atom, then theorem 4.2 along with lemma 2.10 shows that $X_t \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$ is a splitting harmonizable process. Since $Z(d\lambda)$ is atomic, $F_X(d\lambda, d\lambda')$ is also; it is concentrated on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ and hence is stationary. If X_t is a splitting stationary process of full splitting rank, it must arise from an atomic vector measure $Z(\lambda)$. However, not all splitting stationary processes arise from atomic vector measures.

Example 4.12 Let $W \in [\mathbf{L}_0^2(P)]^2$ be such that $\mathbf{E}(WW^*) = \mathbf{I}_2$ and let $\lambda_1, \lambda_2 \in \hat{\mathbf{D}}$ with $\lambda_1 \neq \lambda_2$. Let $H_1(d\lambda)$ and $H_2(d\lambda)$ be two atomic 2×2 measures with support on λ_1 and λ_2 respectively. Let

$$H_1(\{\lambda_1\}) \stackrel{\text{def}}{=} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } H_2(\{\lambda_2\}) \stackrel{\text{def}}{=} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

Notice that $H_1(\{\lambda_1\})H_2^*(\{\lambda_2\}) = H_2(\{\lambda_2\})H_1^*(\{\lambda_1\}) = \mathbf{0}_2$. Let $X_t \stackrel{\text{def}}{=} (\hat{H}_1(t) + \hat{H}_2(t))W$. The spectral measure $F(d\lambda, d\lambda')$ of X_t is concentrated on (λ_1, λ_1) and (λ_2, λ_2), so X_t is stationary. Given $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$, the vector measure $Z(d\lambda)$ can not be atomic since the spectral measure is not. Theorem 4.2 implies X_t is splitting, yet theorem 4.11 shows that X_t can not have full splitting rank (or else the spectral measure of X_t would have support at (λ_1, λ_2)).

The requirement that a process be both stationary and splitting is a strong one since $r(t-s) = H(s)H^*(t)$ relates the product of two matrix valued functions to the difference in their arguments.

4.2 Density of the Set of Splitting Processes

Finite linear combinations of weakly harmonizable processes are weakly harmonizable since if $X_t^j = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_j(d\lambda)$, then

$$\sum_{j=1}^{n} a_j X_t^j = \int_{\hat{\mathbf{D}}} e^{it\lambda} \sum_{j=1}^{n} a_j Z_j(d\lambda).$$

Theorem 4.2 implies that finite linear combinations of splitting processes are still splitting. Thus the set of splitting harmonizable processes is a vector space. In the continuous parameter case, D. Dehay and R. Moché [4] showed that the closure of strongly harmonizable processes in the compact convergence topology contains all continuous, bounded random processes. Therefore it is necessary to introduce finer topologies as follows.

Definition 4.13 Given weakly harmonizable processes, X_t^j and X_t, with the spectral representations $X_t^j = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_j(d\lambda)$ and $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$, the processes X_t^j are said to converge to X_t in the *frequency topology* iff $\text{l.i.m.}_{j\uparrow\infty} Z_j(\Delta) = Z(\Delta)$ for every $\Delta \in \mathcal{B}$.

If $Z_j(\Delta) \to Z(\Delta)$ in $[\mathbf{L}_0^2(P)]^n$ for every $\Delta \in \mathcal{B}$, then $Z(\cdot)$ is a vector measure as a consequence of the Vitali-Hahn-Saks theorem [6, IV.10.6].

Theorem 4.14 *The set of splitting harmonizable processes is dense in the set of weakly harmonizable processes with respect to the frequency topology.*

Proof: Let $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$ be an n-dimensional harmonizable process and let $\{W_k\}_{k\in\mathbf{Z}}$ be an orthonormal basis for $H_Z^-(\hat{\mathbf{D}})$. For $k \in \mathbf{Z}^+$ let π_k be the orthogonal projection of $H_Z^-(\hat{\mathbf{D}})$ onto $\text{sp}\{W_1, \dots, W_k\}$. Define $Z_k(\cdot) \stackrel{\text{def}}{=} (\pi_k Z^1(\cdot), \dots, \pi_k Z^n(\cdot))$. We thus have $\text{l.i.m.}_{k\uparrow\infty} Z_k(\Delta) = Z(\Delta)$ for each $\Delta \in \mathcal{B}$.

It remains to show that $X_k \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_k(d\lambda)$ is a splitting harmonizable processes. It certainly is harmonizable (and if it is splitting it will be necessarily strongly harmonizable). However $\dim \mathbf{H}_{X_k}^-(\infty) = \mathbf{H}_{Z_k}^-(\hat{\mathbf{D}})$ by lemma 2.10 and $\mathbf{H}_{Z_k}^-(\infty) \le nk$, so, by theorem 4.2, X_k is splitting. ∎

A related topology using only covariance functions is given by:

Definition 4.15 Given weakly harmonizable processes, X_t^j and X_t, with spectral bimeasures $F_j(d\lambda, d\lambda')$ and $F(d\lambda, d\lambda')$ respectively, the processes X_t^j converge to X_t in the *bimeasure topology* iff $\lim_{j\uparrow\infty} F_j(\Delta, \Delta') = F(\Delta, \Delta')$ for every $\Delta, \Delta' \in \mathcal{B}$.

Theorem 4.16 *Convergence in the frequency topology implies convergence in the bimeasure topology. In particular, the set of splitting harmonizable processes is dense in the set of weakly harmonizable processes with respect to the bimeasure topology.*

Proof: Let $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z(d\lambda)$ be a weakly harmonizable process. By theorem 4.14 there exists a sequence of splitting harmonizable processes, $\{X_t^j = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_j(d\lambda)\}_{j=1}^{\infty}$, that converges to X_t in the frequency topology. Then

$$F_j(\Delta, \Delta') = \mathbf{E}(Z_j(\Delta) Z_j^*(\Delta')) \to \mathbf{E}(Z(\Delta) Z^*(\Delta')) = F(\Delta, \Delta'). \qquad ∎$$

We also introduce the following concept to establish the next theorem on this subject.

Definition 4.17 The *weak-star topology* on the set of complex Radon vector measures on a topological measure space $\hat{\mathbf{D}}$ is the weakest topology such that if $Z_j(\cdot)$ and $Z(\cdot)$ are complex Radon vector measures, $j \in \mathbf{Z}^+$, then $Z_j \to Z$ iff for every bounded, continuous function $f(\cdot)$ on $\hat{\mathbf{D}}$,

$$\lim_{j\uparrow\infty} \int_{\hat{\mathbf{D}}} f(\lambda)\, Z_j(d\lambda) = \int_{\hat{\mathbf{D}}} f(\lambda)\, Z(d\lambda).$$

Similarly, the *weak-star topology* on the set of matrix valued Radon measures on the topological measure space $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ is the weakest topology such that if $F_j(\cdot,\cdot)$ and $F(\cdot,\cdot)$ are matrix valued Radon measures, $j \in \mathbf{Z}^+$, then $F_j \to F$ iff for every bounded, continuous functions $f(\cdot)$ and $g(\cdot)$ on $\hat{\mathbf{D}}$,

$$\lim_{j\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} f(\lambda)g^*(\lambda')\, F_j(d\lambda, d\lambda') = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} f(\lambda)g^*(\lambda')\, F(d\lambda, d\lambda').$$

The following lemma will be used in the proof of the desired result.

Lemma 4.18 *Molecular vector measures are dense in the weak-star topology on the set of complex Radon vector measures on* $\hat{\mathbf{D}}$.

Proof: Consider the case of $\mathbf{T}$ (the $\mathbf{R}$ case is similar). Let $Z(d\lambda)$ be a vector measure taking values on $[\mathbf{L}_0^2(P)]^p$. For each $n \in \mathbf{Z}^+$ and $-2^n \le j < 2^n$ define the sets $\Gamma_j^n \stackrel{\text{def}}{=} \left[\frac{\pi j}{2^n}, \frac{\pi(j+1)}{2^n}\right)$. Notice that for fixed n, $\Gamma_j^n \cap \Gamma_k^n = \emptyset$ iff $j \neq k$ and that $\cup_{j=-2^n}^{2^n-1} \Gamma_j^n = \mathbf{T}$. Define the molecular vector measures Z_n so that

1. $Z_n\left(\left\{\frac{\pi j}{2^n}\right\}\right) = Z(\Gamma_j^n)$ for $-2^n \le j < 2^n$ and
2. $|Z_n|\left(\mathbf{T} - \left\{\frac{\pi j}{2^n} : -2^n \le j < 2^n\right\}\right) = 0.$

Choose $\epsilon > 0$ and fix $f(\cdot)$, an arbitrary continuous bounded function on $\mathbf{T}$. Choose $N_f \in \mathbf{Z}^+$ so that $n > N_f$ implies $\sup\{|f(x) - f(y)| : x, y \in \Gamma_j^n,\ -2^n \le j < 2^n\} < \frac{\epsilon}{\|Z\|(\mathbf{T})}$ where $\|Z\|$ is the semivariation of Z (see [5]). Thus for $n > N_f$ and $x^* \in \left([\mathbf{L}_0^2(P)]^p\right)^*$ with $\|x^*\|_{[\mathbf{L}_0^2(P)]^p} \le 1$ (identifying the Hilbert space and its adjoint),

$$\begin{aligned} 0 &\le \left| x^*\left(\int_{\mathbf{T}} f(\lambda)\, Z_n(d\lambda) - \int_{\mathbf{T}} f(\lambda)\, Z(d\lambda)\right)\right| \\ &\le \sum_{j=-2^n}^{2^n-1} \left| \int_{\Gamma_j^n} f(\lambda)\,(x^* \circ Z_n)(d\lambda) - \int_{\Gamma_j^n} f(\lambda)\,(x^* \circ Z)(d\lambda)\right| \end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=-2^n}^{2^n-1} \left| f\left(\frac{\pi j}{2^n}\right)(x^* \circ Z)(\Gamma_j^n) - \int_{\Gamma_j^n} f(\lambda)\,(x^* \circ Z)(d\lambda) \right| \\
&\le \sum_{j=-2^n}^{2^n-1} \int_{\Gamma_j^n} \left| f\left(\frac{\pi j}{2^n}\right) - f(\lambda) \right| |x^* \circ Z|(d\lambda) \\
&\le \sum_{j=-2^n}^{2^n-1} \frac{\epsilon}{\|Z\|(\mathbf{T})} \int_{\Gamma_j^n} |x^* \circ Z|(d\lambda) = \epsilon
\end{aligned}
$$

because $\|Z\|(\mathbf{T}) \stackrel{\text{def}}{=} \sup\{|x^* Z|(\mathbf{T}) : x^* \in \left([\mathbf{L}_0^2(P)]^p\right)^*, \|x^*\|_{[\mathbf{L}_0^2(P)]^p} \le 1\}$. Hence for $n > N_f$ one has $|\int_{\mathbf{T}} f(\lambda)\, Z_n(d\lambda) - \int_{\mathbf{T}} f(\lambda)\, Z(d\lambda)|_{[\mathbf{L}_0^2(P)]^p} \le \epsilon$. Thus $Z_n \to Z$ in the weak-star topology. ∎

Theorem 4.19 *The set of spectral measures of strongly harmonizable processes is contained in the weak-star closure of the set of the spectral measures of splitting harmonizable processes.*

Proof: Let $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_X(d\lambda)$ be a strongly harmonizable process with $Z_n(\cdot) \to Z(\cdot)$ in the weak-star topology where $\{Z_n\}_{n\in\mathbf{Z}^+}$ is a sequence of molecular vector measures constructed as in the proof of lemma 4.18. Given the harmonizable processes, $X_t^n \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_n(d\lambda)$, lemma 2.10 implies $\dim \mathbf{H}^-_{X^n}(\infty) = \dim \mathbf{H}^-_{Z_n}(\infty) < \infty$. Theorem 4.2 shows that X_t^n are splitting harmonizable processes. Let $F_n(\cdot,\cdot)$ be the spectral measure of X_t^n and $f(\cdot)$ and $g(\cdot)$ be bounded continuous functions on $\hat{\mathbf{D}}$, then

$$
\begin{aligned}
\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} f(\lambda) g^*(\lambda')\, F_n(d\lambda, d\lambda') &= \mathbf{E}\left(\int_{\hat{\mathbf{D}}} f(\lambda)\, Z_n(d\lambda) \left(\int_{\hat{\mathbf{D}}} g(\lambda')\, Z_n(d\lambda')\right)^*\right) \\
&\to \mathbf{E}\left(\int_{\hat{\mathbf{D}}} f(\lambda)\, Z(d\lambda) \left(\int_{\hat{\mathbf{D}}} g(\lambda')\, Z(d\lambda')\right)^*\right) \\
&= \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} f(\lambda) g^*(\lambda')\, F(d\lambda, d\lambda'). \quad ∎
\end{aligned}
$$

5 FACTORIZABLE SPECTRAL MEASURES

We now analyze a subclass of strongly harmonizable processes.

Definition 5.1 An n-dimensional weakly or strongly harmonizable process, X_t, has *factorizable spectral measure (f.s.m.)* iff its covariance function can be represented as

$$
r(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} c(\lambda) c^*(\lambda')\, \mu(d\lambda, d\lambda') \tag{5.9}
$$

where $c(\cdot)$ is an $n \times p$ matrix valued function with components in $\mathbf{L}^2(d\lambda)$. A weakly or strongly harmonizable process with f.s.m. is called an *f.s.m. process*[5].

For strongly harmonizable f.s.m. processes, one can conclude from

$$\begin{aligned} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} c(\lambda) c^*(\lambda')\, \mu(d\lambda, d\lambda') &= r(s,t) = r^*(t,s) \\ &= \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} c(\lambda') c^*(\lambda)\, \bar{\mu}(d\lambda', d\lambda) \end{aligned}$$

and the uniqueness of characteristic functions, that the controlling measure satisfies $\mu(\Delta, \Delta') = \bar{\mu}(\Delta', \Delta)$ for $\Delta, \Delta' \subseteq \{\lambda : c(\lambda)c^*(\lambda) \neq \mathbf{0}_n\}$. However, not all strongly harmonizable processes are f.s.m. as the following example shows.

Example 5.2 Let

$$\mu(d\lambda) \stackrel{\text{def}}{=} \begin{cases} d\lambda & \text{on } [-1,1] \text{ and} \\ \frac{d\lambda}{\lambda^4} & \text{on } (-\infty,-1) \cup (1,\infty) \end{cases}$$

and let W_t be the Wiener process defined on $[0,\infty)$ with covariance function, $r_W(s,t) = \min\{s,t\}$. For $t \in \mathbf{R}$, define $X_t \stackrel{\text{def}}{=} W_{|t|}$. Note that $r_X(s,t) = \min\{|s|,|t|\}$ is a positive semi-definite kernel by theorem 2.15. Let

$$F(d\lambda, d\lambda') \stackrel{\text{def}}{=} \begin{pmatrix} \min\{|\lambda|,|\lambda'|\} & 0 \\ 0 & 1 \end{pmatrix} \mu(d\lambda)\, \bar{\mu}(d\lambda').$$

Since $\min\{|\cdot|,|\cdot|\}$ is $\mu(d\lambda)\,\bar{\mu}(d\lambda')$ integrable, $F(\cdot,\cdot)$ is a spectral measure by proposition 2.16. Clearly $F(\cdot,\cdot)$ does not split so

$$r(s,t) \stackrel{\text{def}}{=} \iint_{\mathbf{R}\times\mathbf{R}} e^{is\lambda - it\lambda'}\, F(d\lambda, d\lambda')$$

is the covariance of a strongly harmonizable process that is not f.s.m..

Since the spectral measure of a stationary process is positive definite it follows that all stationary processes are f.s.m. processes. Theorem 4.11 shows that all harmonizable splitting processes are f.s.m., though not all f.s.m. process are splitting. The next example is a non-stationary, non-splitting, strongly harmonizable f.s.m. process.

Example 5.3 Let the positive semi-definite kernel, $r_X(\lambda, \lambda') = \min\{|\lambda|,|\lambda'|\}$, and $\mu(d\lambda)$ be as in example 5.2. Proposition 2.16 implies that

$$\min\{|\lambda|,|\lambda'|\}\, \mu(d\lambda)\, \bar{\mu}(d\lambda')$$

[5] J. Kelsh [10] considers only f.s.m. processes when $\mu(d\lambda, d\lambda')$ is a non-negative measure.

is a spectral measure and hence by theorem 2.15 there exists X_t with covariance function:

$$r_X(s,t) = \iint_{\mathbf{R}\times\mathbf{R}} e^{is\lambda - it\lambda'} \min\{\lambda, \lambda'\}\, \mu(d\lambda)\, \bar{\mu}(d\lambda').$$

Thus X_t is a non-stationary strongly harmonizable f.s.m. process (using the notation of (5.9) we have "$\mu(d\lambda, d\lambda') = \min\{\lambda, \lambda'\}\, \mu(d\lambda)\, \bar{\mu}(d\lambda')$" and $c(\cdot) = 1$).

Other examples of non-stationary strongly harmonizable f.s.m. process can be obtained from the work of section 4.

We now analyze a proper subset of the set of f.s.m. processes.

Definition 5.4 The covariance function of an n-dimensional strongly (or weakly) harmonizable f.s.m. process, X_t, has a *virile covariance representation* iff its covariance has a representation (5.9) with

1. $c(\cdot)$ equal to the inverse Fourier transform of its Fourier transform.[6]

2. For $N \in \mathbf{Z}^+$, letting

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \sum_{|j|>N} \hat{c}(j) e^{ij\lambda} & \text{discrete case,} \\ \int_{|\theta|>N} \hat{c}(\theta) e^{i\theta\lambda}\, d\theta & \text{continuous case,} \end{cases}$$

then

$$\lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda) c_N^*(\lambda') \mu(d\lambda, d\lambda') = \mathbf{0}_n.$$

In (5.9), if A is the set where $c(\cdot)$ and the inverse Fourier transform of its Fourier transform differ and if $|\mu|(A, A) = 0$, then one can use the inverse Fourier transform of the Fourier transform instead of the original $c(\cdot)$ in (5.9). Hence condition (1) for a virile covariance representation is satisfied. If a discrete strongly harmonizable f.s.m. process has its spectral measure equal to Lebesgue measure (on $\mathbf{T} \times \mathbf{T}$ or on the diagonal of $\mathbf{T} \times \mathbf{T}$) then it has a virile covariance representation (since condition (2) is true in this case too). In particular, clipped discrete random processes have virile covariance representation (see theorem 4.5).

Lemma 5.5 *Given an m-dimensional strongly harmonizable process, ξ_t, with spectral measure equal to $F_\xi(d\lambda, d\lambda') = \mu_\xi(d\lambda, d\lambda')\mathbf{I}_m$ and an $n \times m$ matrix valued function $c(\cdot)$ such that*

1. $c(\cdot)$ equals the inverse Fourier transform of its Fourier transform and

[6]Since $c(\cdot) \in \mathbf{L}^2(d\lambda)$, Fourier analysis shows that $c(\cdot)$ equals the inverse Fourier transform of its Fourier transform a.e. with respect to Lebesgue measure. Equality everywhere is demanded here.

2. for $N \in \mathbf{Z}^+$, letting

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \sum_{|j|>N} \hat{c}(j)e^{ij\lambda} & \text{discrete case,} \\ \int_{|\theta|>N} \hat{c}(\theta)e^{i\theta\lambda}\,d\theta & \text{continuous case,} \end{cases}$$

one has

$$\lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)c_N^*(\lambda')\mu_\xi(d\lambda, d\lambda') = \mathbf{0}_n.$$

Then

$$\underset{N\uparrow\infty}{\text{l.i.m.}} \int_{\mathbf{T}} \sum_{j=-N}^{N} \hat{c}(j)e^{ij\lambda}\, Z_\xi(d\lambda) = \int_{\mathbf{T}} \sum_{j\in\mathbf{Z}} \hat{c}(j)e^{ij\lambda}\, Z_\xi(d\lambda)$$

in the discrete case and

$$\underset{N\uparrow\infty}{\text{l.i.m.}} \int_{\mathbf{R}} \left(\int_{-N}^{N} \hat{c}(\lambda)e^{i\theta\lambda}\,d\theta\right) Z_\xi(d\lambda) = \int_{\mathbf{R}} \left(\int_{\mathbf{R}} \hat{c}(\lambda)e^{i\theta\lambda}\,d\theta\right) Z_\xi(d\lambda)$$

in the continuous case.

Proof: Let $c_N(\lambda)$ be as in definition 5.4. Using theorem 3.3 it suffices to show that $((c_N(\cdot), c_N(\cdot)))_{\mathbf{L}^2(F_\xi,n)} \downarrow 0$ as $N \uparrow \infty$. However,

$$\begin{aligned} \lim_{N\uparrow\infty} ((c_N(\cdot), c_N(\cdot)))_{\mathbf{L}^2(F_\xi,n)} &= \lim_{N\uparrow\infty} \operatorname{tr} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)\mathbf{E}(Z_\xi(d\lambda)Z_\xi^*(d\lambda'))c_N^*(\lambda') \\ &= \operatorname{tr} \lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)c_N^*(\lambda')\mu_\xi(d\lambda, d\lambda') = 0. \quad \blacksquare \end{aligned}$$

Condition (2) in definition 5.4 relates the function $c(\cdot)$ with measure $\mu(d\lambda, d\lambda')$. Condition (2) is satisfied for any $c(\cdot) \in \mathbf{L}^2(d\lambda)$ if $\mu(d\lambda, d\lambda')$ is one dimensional Lebesgue measure on the diagonal of $\mathbf{T} \times \mathbf{T}$. On the other hand, for the discrete case, if $c(\cdot)$ has an absolutely convergent Fourier series[7], so that

$$\sum_{j\in\mathbf{Z}} |\hat{c}_{k\ell}(j)| < \infty \qquad 1 \le k \le n,\ 1 \le \ell \le m,$$

then condition (2) is satisfied no matter what $\mu(d\lambda, d\lambda')$ is. Between these two extremes, condition (2) depends on both $c(\cdot)$ and $\mu(d\lambda, d\lambda')$.

Not all weakly or strongly harmonizable f.s.m. processes have a virile covariance representation, as the following example shows.

---

[7] Of interest is the following theorem:

Theorem 5.6 (Zygmund) *Let $c(\cdot)$ be of bounded variation on $\mathbf{T}$ and assume $c(\cdot) \in \mathrm{Lip}_\alpha(\mathbf{T})$ for some $\alpha > 0$. Then $c(\cdot)$ has absolutely convergent Fourier series.*

(see [9, section 1.6])

Example 5.7 Let $\mu(d\lambda, d\lambda') \stackrel{\text{def}}{=} d\lambda\, d\lambda' + \delta_{\{0\}}(d\lambda)\,\delta_{\{0\}}(d\lambda')$ where $\delta_{\{0\}}(d\lambda)$ is the Dirac measure with mass one at the point 0. Let $\mathcal{F}(c) \stackrel{\text{def}}{=} \hat{c}$ with

$$c(\lambda) \stackrel{\text{def}}{=} \begin{cases} 0 & \text{if } \lambda \neq 0, \\ 1 & \text{if } \lambda = 0 \end{cases} .$$

Now

$$c(\lambda)c^*(\lambda')\,\mu(d\lambda, d\lambda') = \delta_{\{0\}}(d\lambda)\,\delta_{\{0\}}(d\lambda') \neq \mathcal{F}^{-1}(\hat{c})(\lambda)(\mathcal{F}^{-1}(\hat{c})(\lambda'))^*\,\mu(d\lambda, d\lambda'),$$

since $\mathcal{F}^{-1}(\hat{c})(\lambda) = 0$ for $\lambda \in \hat{\mathbf{D}}$.

For a subclass of f.s.m. processes it is possible to define the concept of "rank" in terms of process' spectral measure.

Definition 5.8 Given an n-dimensional f.s.m. process of rank m, with covariance representation as in (5.9), if $p = m$ ($c(\cdot)$ is an $n \times p$ matrix function), X_t is an *f.s.m. process with full rank m*. An n-dimensional harmonizable f.s.m. process of full rank n is said to have *maximal rank*.

Given an harmonizable splitting process, splitting rank is unrelated to the f.s.m. rank as the following example shows.

Example 5.9 Let $f : \mathbf{T} \to \mathbf{R}$ be defined as

$$f(x) \stackrel{\text{def}}{=} \begin{cases} x & \text{if } x > 0 \text{ and} \\ 0 & \text{otherwise} \end{cases} .$$

Let

$$P(\lambda) \stackrel{\text{def}}{=} \begin{pmatrix} f(\lambda) & 0 & 0 \\ 0 & f(\lambda) & 0 \\ 0 & 0 & f(-\lambda) \end{pmatrix},$$

and let the spectral measure be $F(d\lambda, d\lambda') \stackrel{\text{def}}{=} P(\lambda)P^*(\lambda')\,d\lambda\,d\lambda'$. Then rank for $F(d\lambda, \lambda')$ is not defined (may be one or two), yet the splitting process corresponding to $F(d\lambda, d\lambda')$ has splitting rank 3 since $r(s,t) = \hat{P}(s)\hat{P}^*(t)$ and $\hat{P}(s)$ is of rank 3 for all s.

An example of a splitting process whose covariance function has constant rank, yet which are not splitting with full rank is given in example 5.12.

If X_t is an f.s.m. process with rank m, it need not have an f.s.m. representation with full rank. Indeed, the following example will show that a harmonizable 3-dimensional process can have an f.s.m. representation of rank

2 and yet not have a full rank representation (there is no 3×2 matrix valued function $M(\cdot)$ such that

$$r(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} M(\lambda)M^*(\lambda')\,\mu(d\lambda, d\lambda')$$

where $\mu(d\lambda, d\lambda')$ is a controlling measure).

We will limit our search among f.s.m. processes to those with representation (5.9) with $P(\cdot)$ a step function taking finitely many values. Fourier analysis shows that the covariance function of an n-dimensional f.s.m. process can simultaneously have f.s.m. representations,

$$\begin{aligned} r(s,t) &= \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} P(\lambda)P^*(\lambda')\,\mu_1(d\lambda, d\lambda') \\ &= \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} M(\lambda)M^*(\lambda')\,\mu_2(d\lambda, d\lambda') \end{aligned}$$

iff

$$P(\lambda)P^*(\lambda')\,\mu_1(d\lambda, d\lambda') = M(\lambda)M^*(\lambda')\,\mu_2(d\lambda, d\lambda').$$

Letting $\mu(d\lambda, d\lambda') \stackrel{\text{def}}{=} |\mu_1|(d\lambda, d\lambda') + |\mu_2|(d\lambda, d\lambda')$, the Radon-Nikodým theorem implies that there exist scalar functions $c_1(\cdot,\cdot)$ and $c_2(\cdot,\cdot)$ such that

$$P(\lambda)P^*(\lambda')c_1(\lambda,\lambda')\,\mu(d\lambda, d\lambda') = M(\lambda)M^*(\lambda')c_2(\lambda,\lambda')\,\mu(d\lambda, d\lambda').$$

Define

$$c(\lambda,\lambda') \stackrel{\text{def}}{=} \begin{cases} \frac{c_2(\lambda,\lambda')}{c_1(\lambda,\lambda')} & \text{if } c_1(\lambda,\lambda') \neq 0, \\ 0 & \text{otherwise} \end{cases}.$$

We now have

$$P(\lambda)P^*(\lambda')\,\mu(d\lambda, d\lambda') = M(\lambda)M^*(\lambda')c(\lambda,\lambda')\,\mu(d\lambda, d\lambda').$$

In light of this one may ask:

Question 5.10 *Given $\{P_1, \ldots, P_\ell\}$, $n \times p$ matrices such that $P_jP_k^*$ are all of rank m for $1 \le j,k \le \ell$, does there exist $\{M_1, \ldots, M_\ell\}$, $n \times m$ matrices and C, an $\ell \times \ell$ matrix such that*

$$P_jP_k^* = c_{jk}M_jM_k^*? \qquad 1 \le j,k \le \ell \tag{5.10}$$

The answer to the above question is not positive.

Example 5.11 Letting $n = 3$ and $m = 2$ above and letting

$$P_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix},\ P_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

then a calculation (see [13]) shows there does not exist M_1 and M_2 such that $P_jP_k^* = c_{jk}M_jM_k^*$ for $1 \le j,k \le 2$.

Examples of n-dimensional processes, $n \geq 3$, having a virile covariance representation with rank m where $2 \leq m < n$, yet not having a virile covariance representation with full rank can now be presented.

Example 5.12 Let P_j be as in example 5.11 for $1 \leq j \leq 2$. Define

$$\tilde{P}_j \stackrel{\text{def}}{=} \begin{pmatrix} P_j & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & 0 & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \mathbf{I}_{m-2} \end{pmatrix}.$$

Now let

$$P(t) \stackrel{\text{def}}{=} \begin{cases} \tilde{P}_1 & \text{if } t \in [-1, 0), \\ \tilde{P}_2 & \text{if } t \in (0, 1], \\ 0 & \text{otherwise} \end{cases}$$

and

$$r(s,t) \stackrel{\text{def}}{=} \iint_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} P(\lambda) P^*(\lambda')\, d\lambda\, d\lambda'.$$

Proposition 2.16 shows $r(\cdot, \cdot)$ is the covariance function of a strongly harmonizable f.s.m. process, say X_t. The process has a virile covariance representation since the control measure is Lebesgue measure. One can verify, as in example 5.11, that X_t has f.s.m. rank m, yet does not have a full rank representation.

The following definition and lemma helps to determine when the answer to the question 5.10 is positive:

Definition 5.13 An S-*domain* is any measurable set $S \subset \hat{\mathbf{D}}^2$ such that

$$(\lambda, \lambda') \in S \Leftrightarrow (\lambda, \lambda),\ (\lambda', \lambda) \text{ and } (\lambda', \lambda') \in S.$$

Lemma 5.14 *Let S be an S-domain, $S_1 \subseteq \hat{\mathbf{D}}$ be the first coordinate projection of S into $\hat{\mathbf{D}}$ and $\{P(\lambda)\}_{\lambda \in S_1}$ be a family of $n \times p$ matrices of constant rank m. Then there exists a family of $n \times m$ matrices, $\{M(\lambda)\}_{\lambda \in S_1}$, such that*

$$P(\lambda) P^*(\lambda') = M(\lambda) M(\lambda'), \qquad \forall (\lambda, \lambda') \in S$$

iff

$$P^*(\lambda)(\mathbf{R}^n) = P^*(\lambda')(\mathbf{R}^n), \qquad \forall (\lambda, \lambda') \in S. \tag{5.11}$$

Proof: First we observe that $(\lambda, \lambda') \in S$ implies $\lambda, \lambda' \in S_1$.

($\Rightarrow$) Suppose that there exists $M(\cdot)$ such that $P(\lambda)P^*(\lambda') = M(\lambda)M^*(\lambda')$ for all $(\lambda, \lambda') \in S$. For each fixed $\lambda \in S_1$ we are given that $M^*(\lambda) : \mathbf{R}^n \mapsto \mathbf{R}^m$

is m-dimensional, so it is onto. This shows that given $a_\lambda \in \mathbf{R}^n$ and $(\lambda, \lambda') \in S$, there exists a point $a_{\lambda'} \in \mathbf{R}^n$ such that $M^*(\lambda')a_{\lambda'} = M^*(\lambda)a_\lambda$. Thus

$$\begin{aligned} 0 &= \; < 0, 0 > \; = \; < M^*(\lambda')a_{\lambda'} - M^*(\lambda)a_\lambda, M^*(\lambda')a_{\lambda'} - M^*(\lambda)a_\lambda > \\ &= \; < M^*(\lambda')a_{\lambda'}, M^*(\lambda')a_{\lambda'} > - < M^*(\lambda')a_{\lambda'}, M^*(\lambda)a_\lambda > \\ &\quad - < M^*(\lambda)a_\lambda, M^*(\lambda')a_{\lambda'} > + < M^*(\lambda)a_\lambda, M^*(\lambda)a_\lambda > \\ &= \; < a_{\lambda'}, M(\lambda')M^*(\lambda')a_{\lambda'} > - < a_{\lambda'}, M(\lambda')M^*(\lambda)a_\lambda > \\ &\quad - < a_\lambda, M(\lambda)M^*(\lambda')a_{\lambda'} > + < a_\lambda, M(\lambda)M^*(\lambda)a_\lambda > \\ &= \; < a_{\lambda'}, P(\lambda')P^*(\lambda')a_{\lambda'} > - < a_{\lambda'}, P(\lambda')P^*(\lambda)a_\lambda > \\ &\quad - < a_\lambda, P(\lambda)P^*(\lambda')a_{\lambda'} > + < a_\lambda, P(\lambda)P^*(\lambda)a_\lambda > \\ &= \; < P^*(\lambda')a_{\lambda'} - P^*(\lambda)a_\lambda, P^*(\lambda')a_{\lambda'} - P^*(\lambda)a_\lambda > . \end{aligned}$$

This implies that $P^*(\lambda')a_{\lambda'} = P^*(\lambda)a_\lambda$. Since there exists such $a_{\lambda'} \in \mathbf{R}^n$ for each $a_\lambda \in \mathbf{R}^n$, we have $P^*(\lambda)(\mathbf{R}^n) \subseteq P^*(\lambda')(\mathbf{R}^n)$. An identical argument will show that $P^*(\lambda')(\mathbf{R}^n) \subseteq P^*(\lambda)(\mathbf{R}^n)$. Thus $P^*(\lambda)(\mathbf{R}^n) = P^*(\lambda')(\mathbf{R}^n)$ for all $(\lambda, \lambda') \in S$.

($\Leftarrow$) Suppose $P^*(\lambda)(\mathbf{R}^n) = P^*(\lambda')(\mathbf{R}^n)$ for all $(\lambda, \lambda') \in S$. Consider the set $A \stackrel{\text{def}}{=} P^*(\lambda)(\mathbf{R}^n) \subseteq \mathbf{C}^p$ (the definition is independent of λ).

Claim: A is m-dimensional.

Proof of claim: Fix λ. Let $a_1, \cdots, a_q \in \mathbf{C}^p$ be an orthonormal basis of A (notice $q \geq m$). Define $b_1, \cdots, b_q \in \mathbf{C}^n$ such that $P^*(\lambda)b_j = a_j$ for $1 \leq j \leq q$. Then $b_1, \cdots, b_q$ are linearly independent. If $q > m$ there exists $b = \sum_{i=1}^q c_i b_i \neq 0$ such that $b \perp P(\lambda)P^*(\lambda)(\mathbf{R}^n)$ since dim $P(\lambda)P^*(\lambda)(\mathbf{R}^n) = m$. But then

$$\begin{aligned} 0 &\neq \; < \sum_{i=1}^q c_i a_i, \sum_{j=1}^q c_j a_j > \; = \; < P^*(\lambda)(\sum_{i=1}^q c_i b_i), P^*(\lambda)(\sum_{j=1}^q c_j b_j) > \\ &= \; < b, P(\lambda)P^*(\lambda)b > \; = \; 0, \end{aligned}$$

a contradiction which thus provides a proof of the claim.

Continuing, let $\mathbf{C}^p$ be written as $A \oplus A^\perp$ and let $\pi_1 : \mathbf{C}^p \to \mathbf{C}^m$ be defined as the first coordinate projection, i.e., $\pi_1 : (x, y) \mapsto x$. Then $\pi_1^* : \mathbf{C}^m \to \mathbf{C}^p$ is defined as $\pi_1^* : x \mapsto (x, 0)$. In particular we have $\pi_1^*\pi_1|_A$ is the identity on A. Thus

$$P(\lambda)P^*(\lambda') = P(\lambda)\pi_1^*\pi_1 P^*(\lambda').$$

Letting $M(\lambda) \stackrel{\text{def}}{=} P(\lambda)\pi_1^*$ completes the proof. ∎

The above lemma answers a question more special than that implied by question 5.10. The following example however shows that, using the notation of lemma 5.14, if the assertion in (5.11) is false, it is still possible that there exists $M(\cdot)$ such that (5.10) holds.

Example 5.15 Let

$$P_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \; P_2 = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$$

A simple calculation shows that there does not exist M_1 and M_2 such that $P_i P_j^* = M_i M_j^*$ using the above algebraic lemma. However, letting

$$M_1 = M_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; C = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

it follows that

$$\begin{aligned} P_1 P_1^* &= M_1 M_1^*, \qquad & P_1 P_2^* &= M_1 M_2^*, \\ P_2 P_1^* &= M_2 M_1^*, \qquad & P_2 P_2^* &= 2 M_2 M_2^*. \end{aligned}$$

6 GENERALIZED MOVING AVERAGES

We now generalize the definition of moving averages from that what is commonly used in the literature for stationary processes (the latter moving averages will henceforth be called orthonormal moving averages).

6.1 Moving Averages

Letting $\delta_0(x) \stackrel{\text{def}}{=} \begin{cases} 1 & \text{if } x = 0, \\ 0 & \text{otherwise} \end{cases}$, one has:

Definition 6.1 A *moving average* representation of an n-dimensional random process, X_t, is a representation

$$X_t = \begin{cases} \sum_{j \in \mathbf{Z}} \hat{c}(j - t) \xi_j & \text{discrete case} \\ \int_{\mathbf{R}} \hat{c}(\lambda - t) \xi_\lambda \, d\lambda & \text{continuous case} \end{cases} \tag{6.12}$$

where (the integral being a Bochner integral (see [5])),

1. $\hat{c}(\cdot)$ is the Fourier transform of an $\mathbf{L}^2(d\lambda)$ function $c : \hat{\mathbf{D}} \to \mathcal{M}_{n,m}$ and
2. $r_\xi(s,t) = \rho(s,t)\mathbf{I}_m$ where $\rho(\cdot,\cdot)$ is the covariance function of a one dimensional process.

Furthermore, if

a.) $\mathbf{E}(\xi_s \xi_t^*) = \mathbf{0}_m$ when $s \neq t$,

b.) $\mathbf{E}(\xi_s \xi_t^*) = \delta_0(t-s)\mathbf{I}_m$ (for the discrete case), c.) ξ_j is a stationary process,

d.) ξ_j is a strongly harmonizable process,

e.) ξ_j is a weakly harmonizable process,

then the moving average representation in (6.12) is, respectively, termed

a.) *orthogonal moving average*,

b.) *orthonormal moving average*,

c.) *stationary moving average*,

d.) *strongly harmonizable moving average*,

e.) *weakly harmonizable moving average*.

Notice that the condition $\mathbf{E}(\xi_s \xi_t^*) = \delta_0(t-s)\mathbf{I}_m$ is attainable only in the case X_t is a discrete process.

There are several different choices for a definition of moving average. If the definition is too strong only a limited class of processes will have a moving average, while if the definition is too weak, just about every process will have a moving average. Every discrete process can be represented as $X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ where $\xi_j = X_j$ and $\hat{c}(j) = \chi_{\{0\}}(j)$. However, if X_t is not a one dimensional process, condition (2) in definition 6.1 may not hold, and thus $\sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ may not be a moving average representation of X_t. Furthermore, a moving average representation need not be unique.

Definition 6.2 A moving average representation (6.12) has *rank* p iff $c(\lambda)$ has rank p for every $\lambda \in \mathbf{T}$. A moving average representation has *full rank* m iff it has rank m.

For weakly and strongly harmonizable processes, the following definition is presented:

Definition 6.3 A weakly/strongly harmonizable moving average (6.12) is a *virile moving average* iff ξ_t is weakly/strongly harmonizable with spectral measure $\mu_\xi(d\lambda, d\lambda')\mathbf{I}_m$ and for $N \in \mathbf{Z}^+$, letting

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \sum_{|j|>N} \hat{c}(j)e^{ij\lambda} & \text{discrete case,} \\ \int_{|\theta|>N} \hat{c}(\theta)e^{i\theta\lambda}\, d\theta & \text{continuous case,} \end{cases}$$

then

$$\lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)c_N^*(\lambda')\, \mu_\xi(d\lambda, d\lambda') = \mathbf{0}_n.$$

The remarks following definition 5.4 and lemma 5.5 are pertinent to the above definition.

Lemma 6.4 *Let $X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ be an orthogonal moving average representation of X_t. If $\sup_{j\in\mathbf{Z}} \|\xi_j\| < \infty$, then it is an orthogonal weakly harmonizable virile moving average representation and X_t is a weakly harmonizable process.*

Proof: Assume that the ξ_j's are orthogonal. Let $K \stackrel{\text{def}}{=} \sup_{j\in\mathbf{Z}} \|\xi_j\|$ and let η_j be another random process such that for $j,k \in \mathbf{Z}$:

1. $\mathbf{E}(\eta_j\eta_k^*) = \delta_0(k-j)\mathbf{I}_m$ and
2. $\mathbf{E}(\eta_j\xi_k^*) = \mathbf{0}_m$

(it may be necessary to augment the original probability space to carry also the η_j). Let $\tilde{\xi}_j \stackrel{\text{def}}{=} \xi_j + \sqrt{(K^2 - \|\xi_j\|^2)}\eta_j$. Since $\frac{\tilde{\xi}_j}{K}$ is orthonormal,

$$Y_t \stackrel{\text{def}}{=} K\sum_{j\in\mathbf{Z}} \hat{c}(j-t)\frac{\tilde{\xi}_j}{K} = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\tilde{\xi}_j$$

is stationary. Letting $\pi : \mathbf{H}^-_{\tilde{\xi}}(\infty) \to \mathbf{H}^-_{\xi}(\infty)$ be the orthogonal projection onto $\mathbf{H}^-_X(\infty)$,

$$X_t = [\pi]^n\, Y_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\, [\pi]^m\, \tilde{\xi}_j = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j,$$

which is a weakly harmonizable moving average representation by the dilation theorem 2.12.

Let $c_N(\cdot) \stackrel{\text{def}}{=} \sum_{|j|>N} \hat{c}(j)e^{ij\lambda}$. Given $Y_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\tilde{\xi}_j$, an orthonormal moving average representation of Y_t (for convenience it is assumed that $K=1$), then $\lim_{N\uparrow\infty} \int_{\mathbf{T}} c_N(\lambda)\, Z_{\tilde{\xi}}(d\lambda) = \mathbf{0}_n$, since

$$\lim_{N\uparrow\infty} \mathbf{E}\left(\int_{\mathbf{T}} c_N(\lambda)\, Z_{\tilde{\xi}}(d\lambda)(\int_{\mathbf{T}} c_N(\lambda')\, Z_{\tilde{\xi}}(d\lambda'))^*\right) = \lim_{N\uparrow\infty} \int_{\mathbf{T}} c_N(\lambda)c_N^*(\lambda)d\lambda = \mathbf{0}_n.$$

One has

$$\begin{aligned}\mathbf{0}_n &= [\pi]^n \lim_{N\uparrow\infty} \int_{\mathbf{T}} c_N(\lambda)\, Z_{\tilde{\xi}}(d\lambda) \\ &= \lim_{N\uparrow\infty} \int_{\mathbf{T}} c_N(\lambda)\ [\pi]^m\, Z_{\tilde{\xi}}(d\lambda) = \lim_{N\uparrow\infty} \int_{\mathbf{T}} c_N(\lambda)\, Z_{\xi}(d\lambda).\end{aligned}$$

Thus

$$\begin{aligned}\mathbf{0}_n &= \lim_{N\uparrow\infty} \mathbf{E}\left(\int_{\mathbf{T}} c_N(\lambda)\, Z_{\xi}(d\lambda)(\int_{\mathbf{T}} c_N(\lambda')\, Z_{\xi}(d\lambda'))^*\right) \\ &= \lim_{N\uparrow\infty} \iint_{\mathbf{T}\times\mathbf{T}} c_N(\lambda)\mathbf{E}\left(Z_{\xi}(d\lambda)Z_{\xi}^*(d\lambda')\right) c_N^*(\lambda') \\ &= \lim_{N\uparrow\infty} \iint_{\mathbf{T}\times\mathbf{T}} c_N(\lambda)c_N^*(\lambda')\, \mu_{\xi}(d\lambda, d\lambda').\end{aligned}$$

It now follows that $X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ is an orthogonal weakly harmonizable virile moving average representation of X_t. ∎

The main result for this section follows.

Theorem 6.5 *An n-dimensional strongly harmonizable process, X_t, has a strongly harmonizable virile moving average representation with full rank m,*

$$X_t = \begin{cases} \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j & \textit{discrete case} \\ \int_{\mathbf{R}} \hat{c}(\lambda - t)\xi_\lambda \, d\lambda & \textit{continuous case} \end{cases}$$

where $r_\xi(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'} \mu_\xi(d\lambda, d\lambda')\mathbf{I}_m$, iff it is a strongly harmonizable process with full rank m and has the following full rank virile covariance representation,

$$r_X(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'} c(\lambda)c^*(\lambda')\,\mu_\xi(d\lambda, d\lambda'). \tag{6.13}$$

Furthermore X_t has spectral characteristic $c(\cdot)$ with respect to the process ξ_j, i.e., $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda} c(\lambda)\, Z_\xi(d\lambda)$.

Proof: The discrete case is proved below. The continuous case is quite similar, though the notation differs in places.

($\Rightarrow$) Fix $t \in \mathbf{Z}$. Then

$$\begin{aligned} X_t &= \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j = \sum_{j\in\mathbf{Z}} \hat{c}(j-t) \int_{\mathbf{T}} e^{ij\lambda}\, Z_\xi(d\lambda) \\ &= \operatorname*{l.i.m.}_{N\uparrow\infty} \sum_{j=-N}^{N} \int_{\mathbf{T}} e^{ij\lambda}\hat{c}(j-t)\, Z_\xi(d\lambda) \\ &= \operatorname*{l.i.m.}_{N\uparrow\infty} \int_{\mathbf{T}} \left(\sum_{j=-N}^{N} \hat{c}(j-t)e^{ij\lambda} \right) Z_\xi(d\lambda) \\ &= \int_{\mathbf{T}} \left(\sum_{j\in\mathbf{Z}} \hat{c}(j-t)e^{ij\lambda} \right) Z_\xi(d\lambda), \qquad \text{by lemma 5.5,} \\ &= \int_{\mathbf{T}} e^{it\lambda} \left(\sum_{j\in\mathbf{Z}} \hat{c}(j-t)e^{i(j-t)\lambda} \right) Z_\xi(d\lambda) \\ &= \int_{\mathbf{T}} e^{it\lambda} \left(\sum_{j\in\mathbf{Z}} \hat{c}(j)e^{ij\lambda} \right) Z_\xi(d\lambda) = \int_{\mathbf{T}} e^{it\lambda} c(\lambda)\, Z_\xi(d\lambda). \end{aligned} \tag{6.14}$$

The covariance of X_t is now calculated.

$$\begin{aligned} \mathbf{E}(X_s X_t^*) &= \mathbf{E}\left(\int_{\mathbf{T}} e^{is\lambda} c(\lambda)\, Z_\xi(d\lambda) \left(\int_{\mathbf{T}} e^{it\lambda'} c(\lambda')\, Z_\xi(d\lambda') \right)^* \right) \\ &= \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'} c(\lambda)\mathbf{E}(Z_\xi(d\lambda)Z_\xi^*(d\lambda'))c^*(\lambda') \\ &= \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'} c(\lambda)c^*(\lambda')\,\mu_\xi(d\lambda, d\lambda'). \end{aligned} \tag{6.15}$$

Since $c(\lambda) = \sum_{j \in \mathbf{Z}} \hat{c}(j) e^{i\lambda j}$ has full rank, X_t has a full rank virile covariance representation. The interchange of the expectation operator and the integral sign is easily justified.

($\Leftarrow$) Assume that the covariance function of X_t has the full rank m representation (6.13) where $c(\lambda) = \sum_{j \in \mathbf{Z}} \hat{c}(j) e^{i\lambda j}$. There exists an $m \times n$ matrix function $\Psi(\cdot)$, not necessarily unique, having the property that $\Psi(\lambda)c(\lambda) = \mathbf{I}_m$. Let $X_t = \int_{\mathbf{T}} e^{i\lambda t} \, Z(d\lambda)$ be the spectral representation of X_t where $Z(\cdot) \stackrel{\text{def}}{=} (Z_1(\cdot), \ldots, Z_n(\cdot))$. Define $\Lambda(d\lambda) = (\Lambda_1(d\lambda), \ldots, \Lambda_m(d\lambda))$ where

$$\Lambda_j(\Delta) \stackrel{\text{def}}{=} \int_\Delta \sum_{k=1}^n \Psi_{jk}(\lambda) \, Z_k(d\lambda) = \int_\Delta \left(\Psi Z\right)_j (d\lambda).$$

Now define the m-dimensional process $\xi_t = (\xi_t^1, \ldots, \xi_t^m)$ by $\xi_t^j \stackrel{\text{def}}{=} \int_{\mathbf{T}} e^{i\lambda t} \, \Lambda_j(d\lambda)$. A simple calculation shows that ξ_j is independent of our choice, $\Psi(\cdot)$, of left inverse to $c(\cdot)$. (One can show $\mathbf{E}((\xi_s - \xi_s')(\xi_s - \xi_s')) = 0$ where ξ_s' is obtained by choosing a different inverse, $\Psi'(\cdot)$ of $c(\cdot)$.)

Notice that

$$\begin{aligned}
\mathbf{E}\left(\xi_s \xi_t^*\right) &= \mathbf{E}\left(\left(\int_{\mathbf{T}} e^{i\lambda s} \, \Lambda(d\lambda)\right) \left(\int_{\mathbf{T}} e^{i\lambda' t} \, \Lambda(d\lambda')\right)^*\right) \\
&= \mathbf{E}\left(\iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \, \Lambda(d\lambda)\Lambda^*(d\lambda')\right) \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \, \mathbf{E}\left(\Lambda(d\lambda)\Lambda^*(d\lambda')\right) \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \left[\Psi(\lambda) F(d\lambda, d\lambda') \Psi^*(\lambda')\right] \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \left[\Psi(\lambda) c(\lambda) c^*(\lambda') \Psi^*(\lambda')\right] \, \mu_\xi(d\lambda, d\lambda') \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \mathbf{I}_m \, \mu_\xi(d\lambda, d\lambda').
\end{aligned}$$

In the above, the interchange of $\mathbf{E}(\cdot)$ and the integral is again easily justified. Thus

$$r_\xi(s,t) = \iint_{\mathbf{T}\times\mathbf{T}} e^{i\lambda s - i\lambda' t} \, \mu_\xi(d\lambda, d\lambda') \mathbf{I}_m.$$

A routine calculation shows that $((c(\lambda)\Psi(\lambda) - \mathbf{I}_n, c(\lambda')\Psi(\lambda') - \mathbf{I}_n)) = 0$. Thus $c(\lambda)\Psi(\lambda) = \mathbf{I}_n$ in $\mathbf{L}^2(F_X, n)$. Using the isomorphism of theorem 3.3, one obtains a strongly harmonizable moving average of full rank for X_t through the following calculation:

$$\begin{aligned}
X_t &= \int_{\mathbf{T}} e^{i\lambda t} \, Z(d\lambda) = \int_{\mathbf{T}} e^{i\lambda t} c(\lambda) \Psi(\lambda) \, Z(d\lambda) \\
&= \int_{\mathbf{T}} e^{i\lambda t} c(\lambda) \, \Lambda(d\lambda)
\end{aligned}$$

$$= \int_{\mathbf{T}} e^{i\lambda t} \sum_{j\in\mathbf{Z}} \hat{c}(j) e^{i\lambda j}\, \Lambda(d\lambda)$$
$$= \sum_{j\in\mathbf{Z}} \hat{c}(j) \int_{\mathbf{T}} e^{i\lambda t} e^{i\lambda j}\, \Lambda(d\lambda), \qquad \text{by lemma 5.5,}$$
$$= \sum_{j\in\mathbf{Z}} \hat{c}(j)\xi_{j+t} = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j. \qquad \blacksquare$$

Lemma 6.6 *If an n-dimensional random process, X_t, has a stationary / strongly harmonizable / weakly harmonizable virile moving average representation (6.12), then it is a stationary / strongly harmonizable / weakly harmonizable process with virile covariance representation (6.13). Conversely, if (6.12) is a virile moving average representation, $c(\cdot)$ has full rank, and X_t is a strongly harmonizable (stationary) process then, the moving average representation is strongly harmonizable (stationary).*

Proof: The discrete case is considered here (the proof of the continuous case is similar).

Assume that X_t is a discrete process with a weakly harmonizable virile moving average representation $X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ and with $\xi_j = \int_{\mathbf{T}} e^{ij\lambda}\, Z_\xi(d\lambda)$. Virility implies that (see the the calculation is deriving (6.14) and (6.15))

$$r_X(s,t) = \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'} c(\lambda) F_\xi(d\lambda, d\lambda') c^*(\lambda'),$$

where $F_\xi(d\lambda, d\lambda')$ is the spectral bimeasure of ξ_j. Clearly, X_t is weakly harmonizable. The lemma is now a consequence of:

$X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ is a strongly harmonizable (stationary) moving average representation.

$\Longleftrightarrow$ ξ_j is a strongly harmonizable (stationary) process.

$\Longleftrightarrow$ $F_\xi(d\lambda, d\lambda')$ is a measure (concentrated on the diagonal of $\mathbf{T}\times\mathbf{T}$).

$\Longrightarrow$ The spectral bimeasure of X_t, which is $c(\lambda)F_\xi(d\lambda, d\lambda')c^*(\lambda')$, is a measure (concentrated on the diagonal of $\mathbf{T}\times\mathbf{T}$). (The $\Leftarrow$ direction follows if the spectral characteristic, $c(\cdot)$, has full rank.)

$\Longleftrightarrow$ X_t is a strongly harmonizable (stationary) process. $\blacksquare$

While virility played a key role in the proof above, the author is not presently aware of an example of a non-virile moving average representation for which the above lemma does not hold.

6.2 Stationary Case

"Stationary orthonormal moving average" is redundant since every orthonormal moving average must be a stationary moving average. If X_t has an orthonormal moving average representation (6.12), then X_t is a stationary process by lemmas 6.4 and 6.6. Furthermore, "orthogonal stationary moving averages" are equivalent (modulo a multiplicative constant) to orthonormal moving averages.

Stationary moving averages are also strongly harmonizable moving averages and hence theorem 6.5 applies. Since the spectral density (with respect to a control measure) of a stationary process is a positive semi-definite function, all stationary processes have covariances with f.s.m. representation. It is also clear that a stationary process has a covariance with full rank f.s.m. representation iff the spectral density (with respect to its control measure) has constant rank. All one dimensional stationary processes thus have a covariance with full rank f.s.m. representation (let the control measure be zero when the spectral density is zero).

Corresponding to theorem 6.5 one has

Theorem 6.7 *An n-dimensional stationary process has a stationary moving average representation with full rank iff it has a full rank virile covariance representation.*

Consider the special case when a stationary process has control measure equal to Lebesgue measure. In this case all f.s.m. covariance representations are virile covariance representations. The above theorem reduces to a classical result [18, theorem 1.9.1].

Theorem 6.8 *An n-dimensional discrete stationary process, X_t, has an orthonormal moving average representation iff its covariance function, $r_X(\cdot)$, has a constant rank representation (i.e., $f(\cdot)$ has constant rank) given by,*

$$r_X(s) = \int_{\mathbf{T}} e^{is\lambda} f(\lambda)\, d\lambda. \tag{6.16}$$

Proof: ($\Rightarrow$) Let $X_t = \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j$ where ξ_j is an orthonormal process. Then $r_\xi(s,t) = \int_{\mathbf{T}} e^{i(s-t)\lambda}\, d\lambda$ and

$$\begin{aligned}
\mathbf{E}(X_s X_t^*) &= \mathbf{E}\left(\int_{\mathbf{T}} e^{is\lambda} c(\lambda)\, Z_\xi(d\lambda) \left(\int_{\mathbf{T}} e^{it\lambda'} c(\lambda')\, Z_\xi(d\lambda')\right)^*\right) \\
&= \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'} c(\lambda)\mathbf{E}(Z_\xi(d\lambda) Z_\xi^*(d\lambda')) c^*(\lambda') \\
&= \int_{\mathbf{T}} e^{i(s-t)\lambda} c(\lambda) c^*(\lambda)\, d\lambda.
\end{aligned}$$

Letting $f(\lambda) \stackrel{\text{def}}{=} c(\lambda)c^*(\lambda)$ we have (6.16).

($\Leftarrow$) Let m be the rank of $f(\cdot)$ and let $c(\cdot)$ be a family of $n \times m$ matrix valued functions on $\mathbf{T}$ such that $c(\lambda)c^*(\lambda) = f(\lambda)$. Furthermore, let $\mathcal{D} \stackrel{\text{def}}{=} \{(\lambda, \lambda') \in \mathbf{T}\times\mathbf{T} : \lambda = \lambda'\}$ and let $\mu_\xi(d\lambda, d\lambda') \stackrel{\text{def}}{=} \chi_{\mathcal{D}}(\lambda, \lambda')\, d\lambda$. Then (6.16) becomes

$$r_X(s,t) = \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'} c(\lambda) c^*(\lambda')\, \mu_\xi(d\lambda, d\lambda').$$

Applying theorem 6.5 we get a strongly harmonizable virile moving average representation with full rank m, namely

$$X_t = \sum_{j \in \mathbf{Z}} \hat{c}(j-t)\xi_j.$$

Since

$$r_\xi(s,t) = \iint_{\mathbf{T}\times\mathbf{T}} e^{is\lambda - it\lambda'}\, \mu_\xi(d\lambda, d\lambda') = \int_{\mathbf{T}} e^{i(s-t)\lambda}\, d\lambda,$$

ξ_j is an orthonormal process and hence X_t has an orthonormal moving average. ∎

7 ONE-SIDED MOVING AVERAGES

In this section we examine some concepts which are also useful in prediction theory.

Definition 7.1 An n-dimensional random process, X_t, has a *one-sided moving average representation* iff it admits the representation of definition 6.1 with $\hat{c}(u) = 0$ for $u > 0$:

$$X_t = \begin{cases} \sum_{j=-\infty}^{t} \hat{c}(j-t)\xi_j & \text{discrete case} \\ \int_{-\infty}^{t} \hat{c}(\lambda - t)\xi_\lambda\, d\lambda & \text{continuous case} \end{cases}. \tag{7.17}$$

A *one-sided strongly/weakly harmonizable (stationary) moving average representation* is one in which ξ_t is a strongly/weakly harmonizable (stationary) process in (7.17).

We can assume that $\hat{c}(0) \neq 0$ in any one-sided moving average. If not, let $k \stackrel{\text{def}}{=} \min\{j \in \mathbf{Z} : \hat{c}(-j) \neq 0\} > 0$. Let $a(\lambda) \stackrel{\text{def}}{=} e^{ik\lambda}c(\lambda)$ and $\tilde{\xi}_j \stackrel{\text{def}}{=} \xi_{j-k}$. Then $\hat{a}(0) \neq 0$ and

$$X_t = \sum_{j=-\infty}^{t-k} \hat{c}(j-t)\xi_j = \sum_{j=-\infty}^{t-k} \hat{a}(j+k-t)\tilde{\xi}_{j+k} = \sum_{j=-\infty}^{t} \hat{a}(j-t)\tilde{\xi}_j.$$

If ξ_j is strongly/weakly harmonizable/stationary then so is $\tilde{\xi}_j$, since $\tilde{\xi}_j = \mathbf{S}^k \xi_j$ where $\mathbf{S}$ is a unit index shift.

Let $\mathbf{H}^2(\hat{\mathbf{D}})$ denote the square integrable Hardy space on $\hat{\mathbf{D}}$ and let $P_r(\cdot)$ denote the Poisson kernel in the discrete case (see [7, 9]). Thus if $\hat{\mathbf{D}} = \mathbf{T}$,

$$P_r(\theta) \stackrel{\text{def}}{=} \sum_{j=-\infty}^{\infty} r^{|j|}e^{ij\theta} = \frac{1-r^2}{1-2r\cos\theta + r^2} = \operatorname{Re}\left(\frac{1+re^{i\theta}}{1-re^{i\theta}}\right).$$

When considering one-sided strongly harmonizable moving averages of full rank the following theorem is useful.

Theorem 7.2 *An n-dimensional discrete (continuous) strongly harmonizable process has a one-sided strongly harmonizable virile moving average with full rank iff it has a virile covariance representation with full rank,*

$$\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'} c(\lambda)c^*(\lambda')\,\mu(d\lambda, d\lambda'),$$

where $c(-\cdot) \in \mathbf{H}^2(\mathbf{T})$ $(c(i\cdot) \in \mathbf{H}^2(\mathbf{R}))$.

Proof: We consider the discrete case first. Note that $c(-\cdot) \in \mathbf{L}^2(d\lambda)$.

By definition, $c(-\cdot) \in \mathbf{H}^2(\mathbf{T})$ iff there exists an analytic function $g(\cdot)$ on the complex open unit ball such that $\text{l.i.m.}_{\rho\uparrow 1}\, g(\rho e^{i\theta}) = c(-\theta)$ (w.r.t. Lebesgue measure – see [7, page 39]). Letting $z = re^{i\theta}$ and letting the power series representation of $g(\cdot)$ be $g(z) = \sum_{n=0}^{\infty} a_n z^n$ we have (see [19, section 5.24])

$$\begin{aligned} g(z) = \sum_{n=0}^{\infty} a_n r^n e^{in\theta} &= \int_{-\pi}^{\pi} c(-e^{it}) P_r(\theta - t)\, dt \\ &= \sum_{n=-\infty}^{\infty} \hat{b}(n) r^{|n|} e^{in\theta} = \sum_{n=-\infty}^{\infty} \hat{c}(-n) r^{|n|} e^{in\theta}, \end{aligned}$$

where $\hat{b}(\cdot)$ are the Fourier coefficients of $c(-\cdot)$. Thus we see that $c(\cdot)$ has its Fourier coefficients with positive index vanish iff $c(-\cdot) \in \mathbf{H}^2(\mathbf{T})$. We saw in section 6 that the coefficients of a strongly harmonizable moving average correspond to the Fourier coefficients of $c(\cdot)$. The corresponding strongly harmonizable moving average takes the form of (7.17) iff the Fourier coefficients with positive index of $c(\cdot)$ are zero iff the Fourier coefficients with negative index of $c(-\cdot)$ are zero or iff $c(-\cdot) \in \mathbf{H}^2(\mathbf{T})$.

The continuous case follows from the Paley-Wiener Theorem (see [7, page 131]). This theorem says that $c(i\cdot) \in \mathbf{H}^2(\mathbf{R})$ iff letting $f(\omega) \overset{\text{def}}{=} c(i\omega)$ we have

$$f(\omega) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \hat{f}(t) e^{-\omega t}\, dt$$

for some function $\hat{f}(\cdot)$. This is true iff $c(-\lambda) = \frac{1}{\sqrt{2\pi}} \int_0^{\infty} \hat{f}(t) e^{-i\lambda t}\, dt$, which is the same as saying, $c(\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \hat{f}(t) e^{-i\lambda t}\, dt$, i.e., the Fourier coefficients of $c(\cdot)$ vanish on the positive axis. The proof follows now as in the discrete case. ∎

In [18, theorem 1.10.1], Rozanov shows that when the spectral density (with respect to Lebesgue measure) of an n-dimensional discrete (continuous) stationary process is a rational function of $e^{i\lambda}$ (of λ), the stationary process has a one-sided orthonormal moving average representation with full rank. The proof involves using Blaschke products to show that the spectral density can be written as $c(\lambda)c^*(\lambda)$ where $c(-\cdot)$ is an $n \times m$ matrix valued function that is analytic in the unit ball in **C** in the discrete case and the lower half of the complex plane in the continuous case. His proof applies only to a more specialized case, but gives a sharper conclusion.

In the stationary case, a rational spectral density (with respect to Lebesgue measure) is positive definite and thus has many $c(\lambda)c^*(\lambda)$ representations. Thus theorem 6.8 shows that there are lots of orthonormal moving average representations for such a stationary process. In the strongly harmonizable case, theorem 6.5 does not require Lebesgue measure for a control measure. This affords us even more freedom than in the orthonormal moving average case.

Theorem 7.3 *An n-dimensional discrete (continuous) strongly harmonizable process has a one-sided strongly harmonizable virile moving average with full rank iff it has a virile covariance representation of full rank m,*

$$\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} c(\lambda)c^*(\lambda')\,\mu(d\lambda, d\lambda')$$

where $c(\cdot)$ can be extended to a meromorphic function in the complex unit disk (in the upper half of the complex plane).

Proof: ($\Rightarrow$) Let $N(\cdot)$ be the meromorphic extension of $c(\cdot)$ on the open unit disk (the upper half plane). For each $1 \le j \le n$ and $1 \le k \le m$ there exists a polynomial $q_{jk}(\cdot)$ such that $q_{jk}(\cdot)N_{jk}(\cdot)$ is analytic. Define the polynomial $q(\lambda) \stackrel{\text{def}}{=} \prod_{j=1}^{n}\prod_{k=1}^{m} q_{jk}(\lambda)$. Notice that $q(\cdot)N(\cdot)$ is analytic. We now have

$$\begin{aligned}\iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} & e^{i\lambda s - i\lambda' t} c(\lambda)c^*(\lambda')\,\mu(d\lambda, d\lambda') \\ &= \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t}[q(\lambda)c(\lambda)][q(\lambda')c(\lambda')]^* \,\frac{\mu(d\lambda, d\lambda')}{q(\lambda)\overline{q(\lambda')}}. \qquad (7.18)\end{aligned}$$

Letting $\tilde{c}(\lambda) \stackrel{\text{def}}{=} q(\lambda)c(\lambda)$ and $\tilde{\mu}(d\lambda, d\lambda') \stackrel{\text{def}}{=} \frac{\mu(d\lambda, d\lambda')}{q(\lambda)\overline{q(\lambda')}}$ we have

$$(7.18) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{i\lambda s - i\lambda' t} \tilde{c}(\lambda)\tilde{c}^*(\lambda')\,\tilde{\mu}(d\lambda, d\lambda')$$

where $\tilde{c}(\cdot)$ is analytic. Since $q(\cdot)$ has only a finite number of roots and since they are in the unit disk (upper half plane), $q(\cdot)$ is bounded away from zero on $\hat{\mathbf{D}}$. This implies that $\tilde{\mu}(d\lambda, d\lambda')$ has finite Vitali variation. Theorem 7.2 gives a desired one-sided strongly harmonizable moving average representation with full rank.

($\Leftarrow$) We assume a one-sided strongly harmonizable moving average with full rank and referring to the ($\Leftarrow$) proof to theorem 6.5 (again the continuous case is similar) we see that $c(\lambda) = \sum_{j=0}^{\infty} \hat{c}(j)e^{ij\lambda}$. This implies that $c(-\cdot) \in \mathbf{H}^2(\mathbf{T})$ ($c(i\cdot) \in \mathbf{H}^2(\mathbf{R})$ in the continuous case by the Paley-Wiener Theorem) and thus $c(\cdot)$ has an analytic (hence meromorphic) extension to the open unit ball (upper half plane). ∎

Note that the above theorem applies to all strongly harmonizable processes which have "meromorphic full rank virile covariance representations" and thus generalizes [18, theorem 1.10.1] in both the stationary and strongly harmonizable cases.

ACKNOWLEDGEMENTS

The author would like to thank the referee for his careful reading of the submitted work and Professor M. M. Rao for his advice and encouragement. This research was partially supported by ONR Contract No. N00014-84-K-0356.

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