Chapter 3

SOME PROPERTIES OF HARMONIZABLE PROCESSES

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Abstract: Harmonizable processes are considered as Fourier transforms of vector measures. The incremental processes derived from harmonizable processes are examined here. In particular, they are seen to be harmonizable too, and are used to establish results concerning the derivatives and definite integrals of harmonizable processes (both of which turn out to be harmonizable too).

The n^{th} moment of a vector measure is introduced and it is suggested that there may be a theory that connects harmonizable processes to their corresponding vector measures much as characteristic functions are connected to their corresponding probability measures.

Finally, a result concerning the moving averages of harmonizable processes with continuous parameter is updated to include moving average representations of the derivative of these harmonizable processes too.

Harmonizable processes, incremental processes, moving averages:

3.1 INTRODUCTION

Convention 3.1.1 Let (Ω, Σ, P) be a probability space. Let $\mathbf{D}$ represent either $\mathbf{R}$, the reals, or $\mathbf{Z}$, the integers. The unit circle, $\mathbf{T}$, will be thought of as $[-\pi, \pi)$. Note that $\hat{\mathbf{D}}$, the topological dual group of $\mathbf{D}$, is either $\mathbf{R}$ or $\mathbf{T}$, depending on whether $\mathbf{D}$ is $\mathbf{R}$ or $\mathbf{T}$ respectively.

Definition 3.1.2 Denote $\mathbf{L}_0^2(P)$ to be the Hilbert space of all complex valued functions $f \in \mathbf{L}^2(P)$ such that $\mathbf{E}(f) = 0$, where $\mathbf{E}(f) \stackrel{\text{def}}{=} \int_\Omega f(x)\,P(dx)$, the expectation of f.

Definition 3.1.3 An n–dimensional vector measure, $Z : \Sigma \to \left[\mathbf{L}_0^2(P)\right]^n$ has *orthogonal increments* iff $\mathbf{E}(Z(\Delta)Z(\Delta')^*) = \mathbf{0}_n$[1] for all $\Delta, \Delta' \in \Sigma$ such that $\Delta \cap \Delta' = \emptyset$.

Definition 3.1.4 An n–dimensional process, $X : \mathbf{D} \to \left[\mathbf{L}_0^2(P)\right]^n$, is *weakly harmonizable* iff X_t can be written as the following Dunford–Schwartz integral:

$$X_t = \int_{\hat{\mathbf{D}}} e^{i\lambda t}\, Z_X(d\lambda).$$

If $Z_X(d\lambda)$ has orthogonal increments, X_t is *stationary* (stationary in the wide or Khinchine sense).

It is somewhat traditional to define weakly harmonizable and stationary processes in terms of their covariance functions, rather than as Fourier transforms of a vector measure. However, the following Theorem states that such definitions are equivalent to the one above.

Theorem 3.1.5 *A process is harmonizable iff its covariance function is expressible as the Morse–Transue integral*[2]*,*

$$r_X(s,t) = \int_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'}\, F_X(d\lambda, d\lambda'). \tag{3.1}$$

A process is stationary iff its covariance function is expressible as

$$r_X(s,t) = \tilde{r}(t-s) = \int_{\hat{\mathbf{D}}} e^{i(s-t)\lambda}\, F_X(d\lambda).$$

A. Kolmogorov [Kolmogorov 41] proved the above Theorem for the stationary case. The weakly harmonizable case is obtained by M.M. Rao in [Rao 82] and, in a more general case, the above Theorem is derived in [Chang–Rao 86]. The "only if" part of the above Theorem is somewhat easy to see. The "if" part in the weakly harmonizable case is a easy consequence of the Dilation Theorem (see [Chang–Rao 86]) and the fact the above Theorem holds in the stationary case.

Definition 3.1.6 The *spectral bimeasure* of a weakly harmonizable process, X_t, is the positive semi–definite bimeasure, $F_X(\cdot,\cdot)$ in (3.1). A weakly harmonizable random process is *strongly harmonizable* iff its spectral bimeasure extends to a measure on the Borel σ–algebra of $\Sigma \times \Sigma$. All strongly harmonizable process are weakly harmonizable with the integral in (3.1) just an ordinary Lebesgue integral.

[1] Here * denotes the adjoint operator, i.e., the conjugate transpose operator.
[2] For definition of a Morse–Transue integral, see [Chang–Rao 86].

3.2 INCREMENTAL PROCESSES

Definition 3.2.1 Given a process, X_t, and $\tau \in \mathbf{D}$, its *increment process*, $(I_\tau X)_t$, is defined as

$$(I_\tau X)_t \stackrel{\text{def}}{=} X_{t+\tau} - X_t.$$

Theorem 3.2.2 *If X_t is harmonizable (stationary) and $\tau \in \mathbf{D}$ then $(I_\tau X)_t$ is harmonizable (stationary). Furthermore, if $\mathbf{D} = \mathbf{R}$ then*

1. *if $X'_t \stackrel{\text{def}}{=} \lim_{\tau \to 0} \frac{1}{t}(I_\tau X)_t$ exists (here convergence is in the mean square sense), then $X'_t = \int_{\mathbf{R}} e^{it\lambda} i\lambda \, Z_X(d\lambda)$ and hence is harmonizable (stationary).*

2. *$\int_0^s X_t \, dt = \int_{\mathbf{R}} \left(\frac{e^{i\lambda s} - 1}{i\lambda} \right) Z_X(d\lambda)$ is harmonizable also (here the integrand equals s when $\lambda = 0$).*

Proof: If X_t is harmonizable, then

$$(I_\tau X)_t = \int_{\hat{\mathbf{D}}} e^{i(t+\tau)\lambda} \, Z_X(d\lambda) - \int_{\hat{\mathbf{D}}} e^{it\lambda} \, Z_X(d\lambda) = \int_{\hat{\mathbf{D}}} e^{it\lambda}(e^{i\tau\lambda} - 1) \, Z_X(d\lambda).$$

Letting $Z_{I_\tau X}(d\lambda) \stackrel{\text{def}}{=} (e^{i\tau\lambda} - 1) \, Z_X(d\lambda)$, one sees that $(I_\tau X)_t$ is harmonizable too. If X_t is stationary, by observing the covariance function of $(I\tau X)_t$ one sees $(I_\tau X)_t$ is stationary too.

If X_t is harmonizable (stationary) with continuous parameter and X'_t exists, then

$$\begin{aligned} X'_t &= \lim_{\tau \to 0} \frac{1}{\tau}(I_\tau X)_t \\ &= \lim_{\tau \to 0} \int_{\mathbf{R}} e^{it\lambda} \left(\frac{e^{i\tau\lambda} - 1}{\tau} \right) Z_X(d\lambda) \\ &= \int_{\mathbf{R}} e^{it\lambda} \lim_{\tau \to 0} \left(\frac{e^{i\tau\lambda} - 1}{\tau} \right) Z_X(d\lambda) \\ &= \int_{\mathbf{R}} e^{it\lambda} i\lambda \, Z_X(d\lambda) \end{aligned}$$

which is again harmonizable (stationary). One can pass the limit inside the integral sign since the integrand converges uniformly on compact sets.

Similarly, if X_t is harmonizable (stationary) with continuous parameter, then

$$\begin{aligned} \int_0^s X_t \, dt &= \int_0^s \left(\int_{\mathbf{R}} e^{i\lambda t} \, Z_X(d\lambda) \right) dt \\ &= \int_{\mathbf{R}} \left(\int_0^s e^{i\lambda t} \, dt \right) Z_X(d\lambda) \\ &= \int_{\mathbf{R}} \left(\frac{e^{i\lambda s} - 1}{i\lambda} \right) Z_X(d\lambda) \end{aligned} \tag{3.2}$$

where $\frac{e^{i\lambda s}-1}{i\lambda}$ equals s when $\lambda = 0$.[3] Again, switching the order of integration is justified by noticing the integrand converges uniformly on compact sets. ∎

R. Swift [Theorem 2.1, Swift 96] states that (3.2) is the spectral representation of a second order mean square differentiable process with strongly harmonizable increments. This result may be obtained from the above Theorem after noticing that "harmonizable" is not destroyed by integration or differentiation and that the increments of a harmonizable process are harmonizable. In particular, since $X_s = \int_0^s X'_t\,dt + X_0$ one can rewrite (3.2) to obtain

$$X_s = \int_{\mathbf{R}} \left(\frac{e^{i\lambda s} - 1}{i\lambda} \right) Z_{X'}(d\lambda) + X_0.$$

3.3 MOMENTS OF HARMONIZABLE PROCESSES

Definition 3.3.1 The n^{th} *moment* of a vector measure, Z, (if it exists) is

$$M_n(Z) \stackrel{\text{def}}{=} \int_{\hat{\mathbf{D}}} \lambda^n \, Z(d\lambda).$$

The n^{th} *moment* of a harmonizable process, $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda}\, Z_X(d\lambda)$, (if it exists) is $M_n(Z_X)$.

In the discrete parameter case, moments of all orders exist. Furthermore,

$$\begin{aligned} X_t &= \int_{\mathbf{T}} e^{i\lambda t}\, Z_X(d\lambda) \\ &= \int_{\mathbf{T}} \left(\sum_{j=0}^{\infty} \frac{(i\lambda t)^j}{j!} \right) Z_X(d\lambda) \\ &= \sum_{j=0}^{\infty} \frac{(it)^j}{j!} \int_{\mathbf{T}} \lambda^j \, Z_X(d\lambda) \\ &= \sum_{j=0}^{\infty} \left(\frac{i^j}{j!} M_j(Z_X) \right) t^j \end{aligned}$$

For the continuous parameter case, one needs a condition to obtain the analogous result. The following Theorem is not hard to prove (derive the $n = 0$ case first).

[3] One could use $\tilde{X}_t \stackrel{\text{def}}{=} X_t - Z_X(\{0\})$ instead of X_t and notice that $Z_{\tilde{X}}(\{0\}) = 0$. Then

$$\int_0^s X_t\,dt = \int_{\mathbf{R}-\{0\}} \left(\frac{e^{i\lambda s} - 1}{i\lambda} \right) Z_X(d\lambda).$$

Theorem 3.3.2 *Let X_t be a continuous parameter harmonizable process with moments of all orders and assume*

$$\lim_{n\uparrow\infty} \int_{\mathbf{R}} \left(\sum_{j=n}^{\infty} \frac{(i\lambda t)^j}{j!} \right) Z(d\lambda) = 0.$$

Then

$$X_t^{(n)} = \sum_{j=n}^{\infty} \left(\frac{i^j}{(j-n)!} M_j(Z_X) \right) t^{j-n}$$

is a continuous parameter harmonizable process with moments of all orders too.

It maybe possible to employ techniques more similar to Paul Lévy's work with characteristic functions than ordinary Fourier Analysis, to answer questions of inversion and convergence for the continuous parameter case. Harmonizable processes would play the role of characteristic functions and spectral measures would play the role of distributions. For instance, the following Theorem and its proof mimics [Proposition 4.2.6, Rao 84].

Theorem 3.3.3 *If the harmonizable process, X_t, with continuous parameter, has $p \in \mathbf{Z}^+$ derivatives at $t = 0$, then X_t has $2[p/2]$ moments, where $[x]$ is the largest integer not exceeding x. On the other hand, if X_t has $p \in \mathbf{Z}^+$ moments, then X_t is p times continuously differentiable.*

3.4 VIRILE REPRESENTATIONS

Definition 3.4.1 An n-dimensional harmonizable process, X_t, has *factorizable spectral measure (f.s.m.)* iff its covariance function can be represented as

$$r_X(s,t) = \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} e^{is\lambda - it\lambda'} \underbrace{c(\lambda)c^*(\lambda')\,\mu(d\lambda, d\lambda')}_{F_X(d\lambda, d\lambda')}, \tag{3.3}$$

where $c(\cdot)$ is an $n \times m$ matrix valued function with components in $\mathbf{L}^2(d\lambda)$ and $\mu(d\lambda, d\lambda')$ is a one dimensional bimeasure. A process with f.s.m. is an *f.s.m. process*. The f.s.m. covariance representation (3.3) has *full rank* m iff $c(\lambda)c^*(\lambda')$ has matrix rank m a.e. with respect to $\mu(\cdot,\cdot)$. If $n = m$, the f.s.m. covariance representation (3.3) of full rank n has *maximal rank*. An n dimensional f.s.m. process, X_t has a *virile covariance representation*, iff

1. $c(\cdot)$ is equal everywhere to the inverse Fourier transform of its Fourier transform.
2. For $N \in \mathbf{Z}^+$, letting

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \displaystyle\sum_{|j|>N} \hat{c}(j)e^{ij\lambda} & \text{discrete case} \\ \displaystyle\int_{s\geq N} \hat{c}(s)e^{is\lambda}\,ds & \text{cont. case} \end{cases}$$

then

$$\lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)c_N^*(\lambda')\mu(d\lambda, d\lambda') = \mathbf{0}_n.$$

Every stationary process is a f.s.m. process since its spectral measure is positive definite. There exists examples of strongly harmonizable processes that are not f.s.m. processes [Example 5.2, Mehlman 91].

If A is the set where $c(\cdot)$ and the inverse Fourier transform of its Fourier transform differ and if $|\mu|(A, A) = 0$, then one can use the inverse Fourier transform of the Fourier transform instead of the original $c(\cdot)$. In particular, if a strongly harmonizable f.s.m. process has its spectral measure equal to Lebesgue measure (on $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ or on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$) then it has a virile covariance representation.

Rank is not defined for all f.s.m. processes. However, rank (if it exists) is independent of f.s.m. representation. Even if the $c(\lambda)c^*(\lambda')$ has constant rank p, there need not be an f.s.m. covariance representation with full rank p.

Definition 3.4.2 A *moving average representation* of a n-dimensional random process, X_t, is a representation

$$X_t = \begin{cases} \sum_{j\in\mathbf{Z}} \hat{c}(j-t)\xi_j & \text{discrete case} \\ \int_{\mathbf{R}} \hat{c}(\lambda-t)\xi_\lambda \, d\lambda & \text{cont. case} \end{cases} \tag{3.4}$$

where

1. $\hat{c}(\cdot)$ is the Fourier transform of an $\mathbf{L}^2(d\lambda)$ function $c : \hat{\mathbf{D}} \to \mathcal{M}_{n,m}$ where $\mathcal{M}_{n,m}$ are all $n \times m$ matrices and
2. $r_\xi(s,t) = \rho(s,t)\mathbf{I}_m$ where $\rho(\cdot,\cdot)$ is the covariance function of a one dimensional process.

A moving average representation (3.4) has *full rank* m iff $c(\lambda)$ has rank m for all $\lambda \in \mathbf{T}$. A harmonizable moving average is a *virile moving average* iff ξ_t is harmonizable with spectral bimeasure $\mu_\xi(d\lambda, d\lambda')\mathbf{I}_m$ and for $N \in \mathbf{Z}^+$, letting

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \sum_{|j|>N} \hat{c}(j)e^{ij\lambda} & \text{discrete case} \\ \int_{s\geq N} \hat{c}(s)e^{is\lambda}\, ds & \text{cont. case} \end{cases}$$

then

$$\lim_{N\uparrow\infty} \iint_{\hat{\mathbf{D}}\times\hat{\mathbf{D}}} c_N(\lambda)c_N^*(\lambda')\,\mu_\xi(d\lambda, d\lambda') = \mathbf{0}_n.$$

If A is the set where $c(\cdot)$ and the inverse Fourier transform of its Fourier transform differ and if $|\mu|(A, A) = 0$, then one can use the inverse Fourier transform of the Fourier transform instead of the original $c(\cdot)$. In particular, if a strongly harmonizable f.s.m. process has its spectral measure equal to Lebesgue measure (on $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ or on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$) then it has a virile covariance representation.

Theorem 3.4.3 *Let X_t be an n-dimensional, continuous parametered, strongly harmonizable process with a strongly harmonizable virile moving average representation with full rank m,*

$$X_t = \iint_{\mathbf{R}\times\mathbf{R}} e^{is\lambda - it\lambda'} c(\lambda)c^*(\lambda')\, \mu(d\lambda, d\lambda')$$

Assume

$$\iint_{\mathbf{R}\times\mathbf{R}} (\lambda\lambda')^j e^{i\lambda s - i\lambda' t}\, F_X(\lambda, \lambda')$$

exists for all $s, t \in \mathbf{R}$ and some $j \in \{0, 1, 2, \cdots\}$. Then for all $k \leq j$,

$$X_t^{(k)} = \int_{\hat{\mathbf{R}}} (i\lambda)^k e^{i\lambda t}\, Z_X(d\lambda) = \int_{\hat{\mathbf{R}}} (i\lambda)^k e^{i\lambda t} c(\lambda) Z_\xi(d\lambda)$$

exists and is a strongly harmonizable process with a strongly harmonizable virile moving average representation with full rank m,

$$X_t^{(k)} = \int_{\mathbf{R}} \hat{c}(\lambda - t)\xi_\lambda^k\, d\lambda$$

where $r_{\xi^k}(s, t) = \iint_{\mathbf{R}\times\mathbf{R}} e^{is\lambda - it\lambda'}\, (\lambda\lambda')^k \mu(d\lambda, d\lambda')\mathbf{I}_m$.

The above Theorem is proved for the case $k = 0$ in [Theorem 6.5, Mehlman 91]. The above Theorem now follows from part one of Theorem 3.2.2 and the repetitive use of the $k = 0$ case.

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