

Moving Average Representation and Prediction for Multidimensional Harmonizable Processes

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Abstract

The conditions under which a purely nondeterministic, multidimensional stationary process has a fundamental moving average representation are well known. The definition of fundamental moving averages is extended to the harmonizable class of processes and conditions are found that indicate when a purely nondeterministic, multidimensional harmonizable process has a fundamental representation.

1 Introduction

Let (Ω, Σ, P) be a probability space.

Definition 1.1 Let $\mathbf{L}_0^2(P)$ to be all complex valued functions $f \in \mathbf{L}^2(P)$ such that $\mathbf{E}(f) = 0$.

Only processes indexed by either \mathbf{Z} (discrete) or \mathbf{R} (continuous) that take on values in $[\mathbf{L}_0^2(P)]^n$ will be considered here. In order to refer to both the discrete and continuous cases simultaneously, the symbol \mathbf{D} will be used denote the index set \mathbf{Z} in the discrete case and \mathbf{R} in the continuous case. The dual group of the index set \mathbf{Z} is $\mathbf{T} \stackrel{\text{def}}{=} [-\pi, \pi)$ and the dual of the index set \mathbf{R} is \mathbf{R} . The dual of the index set \mathbf{D} will be denoted as $\hat{\mathbf{D}}$.

Definition 1.2 A process, X_t , is *stationary* if and only if the covariance function of X_t is a function of the difference of its arguments, i.e., $r_X(s, t) = r_X(s + u, t + u)$ for all u .

*The author recalls how fortunate he was to have M.M. Rao as his advisor in graduate school. One could not ask for a better mentor.

Definition 1.3 An n -dimensional random process, X_t , is *weakly harmonizable* if and only if its covariance function can be represented as

$$r_X(s, t) = \int_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} e^{is\lambda - it\lambda'} F_X(d\lambda, d\lambda'),$$

where $F_X(\cdot, \cdot)$ is a positive definite bimeasure, referred to as the *spectral bimeasure* of X_t . A harmonizable random process is *strongly harmonizable* if and only if its spectral bimeasure is a measure. Otherwise, the above integral is a Morse–Transue integral [1].

A harmonizable process, X_t , is stationary if and only if $F_X(d\lambda, d\lambda')$ has support on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$, i.e., $r_X(s, t) = \int_{\hat{\mathbf{D}}} e^{i(s-t)\lambda} F_X(d\lambda)$.

2 Spectral Representation

The following theorem was given by A. Kolmogorov [4]:

Theorem 2.1 *An n -dimensional process, X_t , is stationary if and only if it has a spectral representation*

$$X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda} Z(d\lambda),$$

where $Z(\cdot)$ is a vector measure with orthogonal increments¹.

The vector measure, $Z(\cdot)$, can be constructed in the stationary case by a method of H. Cramér [2]; or also by A. Blanc-Lapierre and R. Fortet (see [9] or [5, volume 2, page 149]).

Definition 2.2 Given a probability space (Ω, Σ, P) and taking any other probability space (Ω', Σ', P') , one can “enlarge” (Ω, Σ, P) to an *augmented probability space*, $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$, by letting $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P}) \stackrel{\text{def}}{=} (\Omega \times \Omega', \Sigma \times \Sigma', P \otimes P')$.

Let (Ω, Σ, P) be a probability space and $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ be an augmentation of that probability space. For each $\tilde{\omega} \in \tilde{\Omega}$ one can write $\tilde{\omega} = (\omega, \omega')$ where $\omega \in \Omega$ and $\omega' \in \Omega'$. Given a random process X_t on (Ω, Σ, P) one can identify X_t with a random process \tilde{X}_t on the augmented

¹A vector measure, $Z(\cdot)$ has *orthogonal increments* if and only if for any $\Delta, \Delta' \in \hat{\mathbf{D}}$ such that $\Delta \cap \Delta' = \emptyset$ one has $\mathbf{E}(Z(\Delta)Z^*(\Delta')) = 0$. The integral is in the sense of Dunford-Schwartz (see [3, chapter IV, section 10]).

²For every $A \in \Sigma$ and $A' \in \Sigma'$, $P \otimes P'$ is defined by $P \otimes P'(A \times A') \stackrel{\text{def}}{=} P(A)P'(A')$.

probability space by letting $\tilde{X}_t(\tilde{\omega}) \stackrel{\text{def}}{=} X_t(\omega)$. Since the distributions of X_t and \tilde{X}_t are the same, the two random variables are indistinguishable from a probabilistic point of view.

The following theorem, proved by M. M. Rao [8, theorem 6.1], implicitly uses this identification. The proof (which is not given here) involves using a Grothendieck type inequality.

Theorem 2.3 (Dilation Theorem) *A random process, X_t , is a harmonizable process if and only if it has a stationary dilation (Y_t, π) , i.e., there exists a stationary process, Y_t , on an augmented probability space $(\tilde{\Omega}, \tilde{\Sigma}, \tilde{P})$ along with an orthogonal projection, $\pi : \mathbf{L}_0^2(\tilde{P}) \rightarrow \mathbf{L}_0^2(P)$, (where $\mathbf{L}_0^2(P)$ is considered embedded in $\mathbf{L}_0^2(\tilde{P})$) such that $X_t = \pi Y_t$. (An n -dimensional version also holds.)*

Given the Dilation Theorem and Theorem 2.1, one can immediately obtain the following Theorem characterizing weakly harmonizable processes. Either the Dilation Theorem or Theorem 2.1 can be obtained from the other.

Theorem 2.4 *An n -dimensional process, X_t , is weakly harmonizable if and only if it has a spectral representation*

$$X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda} Z(d\lambda), \quad (1)$$

where $Z(d\lambda)$ is a vector measure, not necessarily of orthogonal increments.

In particular for $a, b \in \hat{\mathbf{D}}$, $a < b$ and $\|Z(\{a\})\|_{[\mathbf{L}_0^2(P)]^n} = \|Z(\{b\})\|_{[\mathbf{L}_0^2(P)]^n} = 0$, one can show that

$$Z([a, b]) \stackrel{\text{def}}{=} \begin{cases} \frac{1}{2\pi} \sum_{k \in \mathbf{Z}} \frac{e^{-iak} - e^{-ibk}}{-ik} X_k & \text{discrete case,} \\ \text{l.i.m.}_{T \uparrow \infty} \frac{1}{2\pi} \int_{-T}^T \frac{e^{-iat} - e^{-ibt}}{-it} X_t dt & \text{continuous case} \end{cases}. \quad (2)$$

The spectral bimeasure, $F(\cdot, \cdot)$, of an n -dimensional harmonizable random process satisfies $F(A, B) = \mathbf{E}(Z(A)Z^*(B))$. In the stationary case $Z(d\lambda)$ is orthogonally scattered so that the spectral measure is concentrated on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ and can be written as $F(d\lambda)$.

Lemma 2.5 *Given an n -dimensional strongly harmonizable process, $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda} Z(d\lambda)$, one has $\mathbf{H}_X^-(\infty) = \mathbf{H}_Z^-(\hat{\mathbf{D}})$.*

Proof: The integral representation of Theorem 2.4 implies $\mathbf{H}_X^-(\infty) \subseteq \mathbf{H}_Z^-(\hat{\mathbf{D}})$. That $\mathbf{H}_Z^-(\hat{\mathbf{D}}) \subseteq \mathbf{H}_X^-(\infty)$ follows from (2). ■

3 Moving Average Representations

Definition 3.1 An n -dimensional strongly harmonizable process, X_t , has *factorizable spectral measure (f.s.m.)* if and only if its covariance function can be represented as

$$r_X(s, t) = \iint_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} e^{is\lambda - it\lambda'} \underbrace{c(\lambda)c^*(\lambda') \mu(d\lambda, d\lambda')}_{F_X(d\lambda, d\lambda')}, \quad (3)$$

where $c(\cdot)$ is an $n \times m$ matrix valued function with components in $\mathbf{L}^2(d\lambda)$ and $\mu(d\lambda, d\lambda')$ is a one dimensional measure. A process with f.s.m. is also called an *f.s.m. process*.

Definition 3.2 A *moving average representation* of an n -dimensional random process, X_t , is a representation

$$X_t = \begin{cases} \sum_{j \in \mathbf{Z}} \hat{c}(j-t) \xi_j & \text{discrete case} \\ \int_{\mathbf{R}} \hat{c}(\lambda-t) \xi_\lambda d\lambda & \text{continuous case} \end{cases} \quad (4)$$

where

1. $\hat{c}(\cdot)$ is the Fourier transform of an $\mathbf{L}^2(d\lambda)$ function $c : \mathbf{T} \rightarrow \mathcal{M}_{n,m}$ where $\mathcal{M}_{n,m}$ is the set all $n \times m$ matrices and
2. $r_\xi(s, t) = \rho(s, t) \mathbf{I}_m$ where $\rho(\cdot, \cdot)$ is the covariance function of a one dimensional process and \mathbf{I}_m is the $m \times m$ identity matrix.

Notice that if $\hat{c}(\lambda) = 0$ for $\lambda > 0$, (4) becomes an *one-sided moving average representation*,

$$X_t = \begin{cases} \sum_{j=-\infty}^t \hat{c}(j-t) \xi_j & \text{discrete case} \\ \int_{-\infty}^t \hat{c}(\lambda-t) \xi_\lambda d\lambda & \text{continuous case} \end{cases} .$$

Definition 3.3 A n -dimensional f.s.m. process with covariance representation (3) has *full rank m covariance representation* if and only if $c(\cdot)$ has rank m . A moving average representation, (where $c : \hat{D} \rightarrow \mathcal{M}_{n,m}$) has *full rank m moving average representation* if and only if $c(\lambda)$ has rank m for all $\lambda \in \hat{D}$.

Definition 3.4 Let X_t be a f.s.m. process with covariance representation (3). For each $N \in \mathbf{Z}^+$ let

$$c_N(\lambda) \stackrel{\text{def}}{=} \begin{cases} \sum_{|j| > N} \hat{c}(j) e^{ij\lambda} & \text{discrete case} \\ \int_{|t| > N} \hat{c}(t) e^{it\lambda} dt & \text{continuous case} \end{cases} .$$

Then (4) is a *virile covariance representation* if and only if ξ_t is harmonizable with spectral bimeasure $\mu_\xi(d\lambda, d\lambda') \mathbf{I}_m$ and

$$\lim_{N \uparrow \infty} \iint_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} c_N(\lambda) c_N^*(\lambda') \mu_\xi(d\lambda, d\lambda') = \mathbf{0}_n$$

where \mathbf{I}_m is the $m \times m$ identity matrix and $\mathbf{0}_n$ is the $n \times n$ zero matrix.

If A is the set where $c(\cdot)$ and the inverse Fourier transform of its Fourier transform differ and if $|\mu|(A, A) = 0$, then one can use the inverse Fourier transform of the Fourier transform instead of the original $c(\cdot)$. In particular, if a strongly harmonizable f.s.m. process has its spectral measure equal to Lebesgue measure (on $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$ or on the diagonal of $\hat{\mathbf{D}} \times \hat{\mathbf{D}}$) then it has a virile covariance representation.

Theorem 3.5 *An n -dimensional strongly harmonizable process, X_t , has a strongly harmonizable virile moving average representation with full rank m ,*

$$X_t = \begin{cases} \sum_{j \in \mathbf{Z}} \hat{c}(j-t) \xi_j & \text{discrete case} \\ \int_{\mathbf{R}} \hat{c}(\lambda-t) \xi_\lambda d\lambda & \text{continuous case} \end{cases}$$

where

$$r_\xi(s, t) = \iint_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} e^{is\lambda - it\lambda'} \mu_\xi(d\lambda, d\lambda') \mathbf{I}_m$$

if and only if it has the following full rank virile covariance representation

$$r_X(s, t) = \iint_{\hat{\mathbf{D}} \times \hat{\mathbf{D}}} e^{is\lambda - it\lambda'} c(\lambda) c^*(\lambda') \mu_\xi(d\lambda, d\lambda').$$

Furthermore $X_t = \int_{\hat{\mathbf{D}}} e^{it\lambda} c(\lambda) Z_\xi(d\lambda)$.

The proof can be found in [6, Theorem 6.5].

4 Fundamental Representations

Definition 4.1 Let $X_t \stackrel{\text{def}}{=} (X_t^{(1)}, \dots, X_t^{(n)})^T$ be a n -dimensional random process. Define

$$\begin{aligned} \mathbf{H}_X^-(t) &\stackrel{\text{def}}{=} \overline{\text{sp}}\{X_s^{(j)} : s \leq t, 1 \leq j \leq n\} \\ \mathbf{H}_X^-(\infty) &\stackrel{\text{def}}{=} \overline{\text{sp}}\{X_s^{(j)} : s \in \mathbf{D}, 1 \leq j \leq n\} \\ \mathbf{H}_X^-(-\infty) &\stackrel{\text{def}}{=} \bigcap_{t \in \mathbf{D}} \mathbf{H}_X^-(t). \end{aligned}$$

where the closure is taken in the $\mathbf{L}_0^2(P)$ sense. The space $H_X^-(\infty)$ is referred to as the *space of observables* of X_t .

Let $Z(\cdot) \stackrel{\text{def}}{=} (Z^{(1)}(\cdot), \dots, Z^{(n)}(\cdot))^T$ be an $[\mathbf{L}_0^2(P)]^n$ valued vector measure. Then for every $\Delta \in \mathcal{B}$ let

$$H_Z^-(\Delta) \stackrel{\text{def}}{=} \overline{\text{sp}}\{Z^{(j)}(\Delta \cap \Delta') : 1 \leq j \leq n, \Delta' \in \mathcal{B}\},$$

where closure is in the $\mathbf{L}_0^2(P)$ sense again. One lets $H_Z^-(t) \stackrel{\text{def}}{=} H_Z^-((-\infty, t])$.

Definition 4.2 A random process, X_t , is *deterministic* if and only if $\mathbf{H}_X^-(-\infty) = \mathbf{H}_X^-(\infty)$. A process that is not deterministic is termed *nondeterministic*. A random process, X_t , is *purely nondeterministic* if and only if $\mathbf{H}_X^-(-\infty) = 0$.

Theorem 4.3 (Wold's Decomposition) *Given a random process, X_t , there exists a unique decomposition*

$$X_t = R_t + S_t$$

where $R_s \perp S_t$, and furthermore R_t is purely nondeterministic and S_t is deterministic. If X_t is weakly harmonizable, R_t and S_t are weakly harmonizable too.

Proof: Let $\pi : H_X^-(\infty) \rightarrow H_X^-(-\infty)$ be the orthogonal projection from $H_X^-(\infty)$ onto $H_X^-(-\infty)$. Then

$$S_t \stackrel{\text{def}}{=} \pi(X_t) = \int_{\mathbf{D}} e^{i\lambda t} \pi(Z_X(d\lambda)) \quad \text{and} \quad R_t \stackrel{\text{def}}{=} X_t - S_t = \int_{\mathbf{D}} e^{i\lambda t} (1 - \pi)(Z_X(d\lambda)). \quad \blacksquare$$

Definition 4.4 Given a purely nondeterministic random process, X_t , with bounded covariance function, a *fundamental representation* of X_t is a representation

$$X_t = \begin{cases} \sum_{j=-\infty}^t \hat{c}(j-t)\xi_j & \text{discrete case} \\ \int_{-\infty}^t \hat{c}(\lambda-t)\xi_\lambda d\lambda & \text{continuous case} \end{cases}$$

such that

$$H_X^-(t) = \begin{cases} H_\xi^-(t) & \text{discrete case} \\ H_\zeta^-(t) & \text{continuous case} \end{cases}$$

where $\zeta(d\lambda) \stackrel{\text{def}}{=} \xi_\lambda d\lambda$ and $t \in \mathbf{D}$. A *strongly/weakly harmonizable (stationary) fundamental representation* is a fundamental representation where ξ_t is strongly/weakly harmonizable (stationary).

There is no reason to believe that a fundamental representation is a one-sided moving representation since there is no claim that $r_\xi(s, t)$ can be written as $\rho(s, t)\mathbf{I}_m$. When a fundamental representation is also a one-sided moving average, it is referred to as a *fundamental moving average representation*. If X_t is one dimensional, then a fundamental representation is also a one-sided moving average representation.

Furthermore, since

$$H_X^-(t) \subseteq \begin{cases} H_\xi^-(t) & \text{discrete case} \\ H_\zeta^-(t) & \text{continuous case} \end{cases} ,$$

a one-sided moving average representation is not generally a fundamental moving average representation.

Definition 4.5 A function $c : \hat{D} \rightarrow \mathcal{M}_{n,m}$ is *maximal* if and only if

1. $c(\cdot) \in \mathbf{H}^2(\hat{D})$ and,
2. if $a(\cdot) \in \mathbf{H}^2(\hat{D})$ and $c(\lambda)c^*(\lambda) = a(\lambda)a^*(\lambda)$ then

$$\begin{aligned} c(0)c^*(0) &\geq a(0)a^*(0) && \text{discrete case} \\ c(i)c^*(i) &\geq a(i)a^*(i) && \text{continuous case} \end{aligned}$$

That is, for the discrete case, the difference $c(0)c^*(0) - a(0)a^*(0)$ is positive definite and $c(0)$ and $a(0)$ are the values at zero of the analytic continuation of $c(\cdot)$ and $a(\cdot)$ respectively to the unit disk in \mathbf{C} . For the continuous case, the difference $c(i)c^*(i) - a(i)a^*(i)$ is positive definite and $c(i)$ and $a(i)$ are the values at i of the analytic continuation of $c(\cdot)$ and $a(\cdot)$ respectively to the upper half plane in \mathbf{C} .

The following Theorem concerns fundamental moving averages, i.e., fundamental representations that are also one-sided moving averages.

Theorem 4.6 (Rozanov) *An n -dimensional stationary process, X_t has a fundamental moving average where $\hat{c}(\cdot)$ is maximal if and only if it is purely nondeterministic.*

Proof: Y. Rozanov established the discrete case (see [10, Theorems II.4.1 and II.4.2]).

For the continuous case, Rozanov establishes a related result [10, Theorems III.2.4 and III.3.1], namely that there exists a representation,

$$X_t = \int_{-\infty}^t \hat{c}(\lambda - t)\zeta(d\lambda) \tag{5}$$

where $\zeta(d\lambda)$ is a m -dimensional vector measure and

1. $c(\cdot)$ is a $n \times m$ full rank matrix valued function that is maximal,
2. $H_X^-(t) = H_{\hat{c}}^-(t)$ for all $t \in \mathbf{R}$, and
3. $\mathbf{E}(\zeta(\Delta)\zeta^*(\Delta')) = m(\Delta \cap \Delta')\mathbf{I}_m$ where $m(d\lambda)$ is Lebesgue measure on \mathbf{R} ,

if and only if X_t is purely nondeterministic.

Suppose the spectral measure, $f_X(\cdot)$ has full rank representation $f_X(\cdot) = c(\cdot)c^*(\cdot)$. Let $\Psi(\cdot)$ be the left inverse of $c(\cdot)$ and define the vector measure

$$\Lambda(\Delta) \stackrel{\text{def}}{=} \int_{\Delta} \Psi(\lambda) Z_X(d\lambda)$$

where Δ is a Borel set of \mathbf{R} . Letting $\Delta = (t_1, t_2)$, Rozanov defines

$$\zeta(\Delta) \stackrel{\text{def}}{=} \int_{\mathbf{R}} \frac{e^{i\lambda t_2} - e^{i\lambda t_1}}{i\lambda} \Lambda(d\lambda).$$

The author defines $\xi_t \stackrel{\text{def}}{=} \int_{\mathbf{R}} e^{i\lambda t} \Lambda(d\lambda)$ in proving Theorem 3.5. Letting $\tilde{\zeta}(d\lambda) \stackrel{\text{def}}{=} \xi_\lambda d\lambda$ one notices that

$$\tilde{\zeta}(\Delta) = \int_{t_1}^{t_2} \xi_\lambda dt = \int_{t_1}^{t_2} \int_{\mathbf{R}} e^{i\lambda t} \Lambda(d\lambda) dt = \int_{\mathbf{R}} \int_{t_1}^{t_2} e^{i\lambda t} dt \Lambda(d\lambda) = \int_{\mathbf{R}} \frac{e^{i\lambda t_2} - e^{i\lambda t_1}}{i\lambda} \Lambda(d\lambda) = \zeta(\Delta).$$

Thus $\xi_\lambda d\lambda = \zeta(d\lambda)$, so the result follows. ■

Definition 4.7 A fundamental representation of a strongly harmonizable process, X_t ,

$$X_t = \begin{cases} \sum_{j=-\infty}^t \hat{a}(j-t) \eta_j & \text{discrete case} \\ \int_{-\infty}^t \hat{a}(\lambda-t) \eta_\lambda d\lambda & \text{continuous case} \end{cases}, \quad (6)$$

dilates to a fundamental moving average of a stationary process if and only if there exists a stationary dilation (Y_t, π) with a stationary fundamental moving average representation,

$$Y_t = \begin{cases} \sum_{j=-\infty}^t \hat{a}(j-t) \tilde{\eta}_j & \text{discrete case} \\ \int_{-\infty}^t \hat{a}(\lambda-t) \tilde{\eta}_\lambda d\lambda & \text{continuous case} \end{cases},$$

where $\pi(\tilde{\eta}) = \eta$.

Theorem 4.8 Suppose an n -dimensional strongly harmonizable, purely nondeterministic process, X_t , has a harmonizable full rank m virile covariance representation (3). Then X_t has a fundamental representation (6) where $\hat{a}(\cdot)$ is the Fourier transform of a maximal $n \times m$ matrix-valued function. Furthermore, the fundamental representation dilates to a fundamental moving average representation of a stationary process.

Proof: By Theorem 3.5, X_t has the representation,

$$X_t = \int_{\mathbf{D}} e^{it\lambda} c(\lambda) Z_\xi(d\lambda).$$

Let $(\tilde{\xi}_\lambda, \pi)$ be a stationary dilation of ξ_λ and let

$$Y_t \stackrel{\text{def}}{=} \int_{\mathbf{D}} e^{it\lambda} c(\lambda) Z_{\tilde{\xi}}(d\lambda). \quad (7)$$

Then $\pi(Y_t) = X_t$. Since Y_t is a linear transformation of $\tilde{\xi}_\lambda$, it is stationary. Thus (Y_t, π) is a stationary dilation of X_t .

Without loss of generality, one can assume that Y_t is purely nondeterministic. If not, let $Y_t = R_t + S_t$ be the Wold decomposition of Y_t and notice that $\pi(H_S^-(\infty)) = \{0\}$ since X_t is purely nondeterministic. Let $\tilde{\pi} : H_Y^-(\infty) \rightarrow H_R^-(\infty)$ be the orthogonal projection of $H_Y^-(\infty)$ onto $H_R^-(\infty)$. Then (R_t, π) and $(\tilde{\pi}\xi, \pi)$ are stationary dilations of X_t and ξ , respectively. One can use $\tilde{\pi}(\tilde{\xi}_t)$ and R_t instead of ξ_t and Y_t .

Since Y_t is a purely nondeterministic stationary process, [10, Theorem III.2.3] gives that Y_t has a full rank p covariance representation. Letting $F_Y(d\lambda)$ denote the spectral measure of Y_t and using (7), one sees that $F_Y(d\lambda) = c(\lambda)F_\xi^-(d\lambda)c^*(\lambda)$. Since $F_\xi^-(d\lambda)$ is a $m \times m$ matrix of measures, the rank of any f.s.m. covariance representation of Y_t must be of rank less than or equal to m . Hence $p \leq m$.

Theorem 4.6 gives that it has a fundamental moving average representation,

$$Y_t = \begin{cases} \sum_{j=-\infty}^t \hat{a}(j-t)\tilde{\eta}_j & \text{discrete case} \\ \int_{-\infty}^t \hat{a}(\lambda-t)\tilde{\eta}_\lambda d\lambda & \text{continuous case} \end{cases} \quad (8)$$

where $\hat{a}(\cdot)$ is the Fourier transform of a maximal $n \times p$ matrix-valued function, $a(\cdot)$, and $H_Y^-(t) = H_{\tilde{\eta}}^-(t)$ ($H_Y^-(t) = H_{\tilde{\zeta}}^-(t)$ where $\tilde{\zeta}(d\lambda) = \tilde{\eta}_\lambda d\lambda$ in the continuous case). Furthermore, Theorem 3.5 gives

$$Y_t = \int_{\mathbf{D}} e^{it\lambda} a(\lambda) Z_{\tilde{\eta}}(d\lambda). \quad (9)$$

Let $\eta \stackrel{\text{def}}{=} \pi(\tilde{\eta})$. Then

$$X_t = \pi(Y_t) = \begin{cases} \sum_{j=-\infty}^t \hat{a}(j-t)\eta_j & \text{discrete case} \\ \int_{-\infty}^t \hat{a}(\lambda-t)\eta_\lambda d\lambda & \text{continuous case} \end{cases} \quad (10)$$

Furthermore, using (9), one has

$$X_t = \pi(Y_t) = \int_{\mathbf{D}} e^{it\lambda} a(\lambda) \pi(Z_{\tilde{\eta}}(d\lambda)) = \int_{\mathbf{D}} e^{it\lambda} a(\lambda) Z_\eta(d\lambda). \quad (11)$$

Since

$$H_X^-(t) = \pi(H_Y^-(t)) = \pi(H_{\tilde{\zeta}}^-(t)) = H_\zeta^-(t),$$

(10) is a fundamental representation of X_t .

Finally, using (11), one has

$$F_X(d\lambda, d\lambda') = a(\lambda)F_\eta(d\lambda, d\lambda')a^*(\lambda').$$

Since $F_X(d\lambda, d\lambda')$ can be represented in a full rank m f.s.m. covariance representation and $F_\eta(d\lambda, d\lambda')$ is a $p \times p$ array of measures with $p \leq m$, one sees that $p = m$. \blacksquare

Finding conditions for a purely nondeterministic process to have a fundamental moving average are more involved. In [7], conditions are given for the discrete case. A one dimensional fundamental representation is always a fundamental moving average representation, but conditions for the existence of a fundamental moving average representation for multi-dimensional, purely nondeterministic, continuous harmonizable processes remains an open question.

In the discrete case, one can always let $c(\lambda) = \mathbf{I}_n$ (where \mathbf{I}_n is the $n \times n$ identity matrix) and thus

$$\hat{c}(j) = \begin{cases} \mathbf{I}_n & \text{if } j = 0 \\ 0 & \text{otherwise} \end{cases}$$

so $X_t = \sum_{j=-\infty}^t \hat{c}(j-t)X_t$ is a fundamental representation of X_t . However, this fundamental representation does not satisfy the requirements of the above Theorem unless X_t has a full rank n f.s.m. covariance representation. In the continuous case, it more difficult to find a fundamental representation, much less one that satisfies the requirements of the above Theorem.

Theorem 4.8 (and its proof) states that if a purely nondeterministic harmonizable process has a full rank virile covariance representation, then it has a “full rank” fundamental representation that dilates to a fundamental moving average representation of a stationary process.

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